Marianne Nolte (Editor)

Including the Highly Gifted and Creative Students – Current Ideas and Future Directions

Proceedings of the 11th International Conference on Mathematical Creativity and Giftedness (MCG 11)

22.08.2019 – 24.08.2019

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INTRODUCTION

The 11th International Conference on Mathematical Creativity and Giftedness (MCG11) took place at the Universität Hamburg, Germany, during August 22th – 24th 2019.

In 1999, Prof. Dr. Hartwig Meissner invited to a conference on mathematical creativity and giftedness at the University of Münster in Germany. This conference became the starting point for an international group of researchers, teachers and practitioners whose shared concerns are on the one hand the development of mathematical creativity and giftedness and on the other hand the exploration of mathematical creativity and giftedness. In the meantime, a series of conference was held in different parts of the world and MCG, the international group for mathematical creativity and giftedness was founded (see www.igmcg.org).

The MCG conference aims at promoting mathematical creativity and giftedness in students of all ages and backgrounds, and supporting interested mathematics educators, mathematicians, psychologists, researchers, teachers and other in this topic interested people.

Nowadays, international exchange of research about creativity and giftedness, of concepts how to foster the development of creativity and high competences in mathematics is highly important. School systems and their development differ from country to country. Thus, we can learn from each other's approaches. We all have in common a rapid change of living conditions. The technical development is progressing rapidly while societies are changing continuously. Today we do not know which will be problems to be solved in the future. We teach our students about subjects, which are important now. Preparing them for an unknown future implies thinking about the cognitive competences but, also about personal traits and openness for changes.

From a societal perspective, we must provide opportunities and environments for individuals can develop adequately. This implies encouragement for creativity. The research about giftedness underlines the impact of environmental factors and the personal interplay with these conditions. We talk about promising children to point out that all students need an education to develop their capability and personality to the fullest. Nevertheless, the conditions of realization of this aim are settled differently in many countries.

We hope that international collaboration and discussion will contribute to preparing our prospective active and responsible members of society to cope well with their responsible tasks in their future professional life.

For further information about the activities of the International Group for Mathematical Creativity and Giftedness (MCG), please visit our website http://www.igmcg.org/home.

Marianne Nolte, Universität Hamburg, Germany, July 2019

MAIN LECTURES

RESEARCH ON MATHEMATICAL GIFTEDNESS IN GERMANY – LOOKING BACK AND AHEAD

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Abstract. *Beginning with William Stern, giftedness research has a long tradition in Germany, whereby in the times of German Partition the developments in East and West were quite different. In addition to psychological studies, didactic research on mathematical giftedness has also been on the rise for about 40 years. The lecture presents important developments and results, especially of mathematics education research in Germany, in their interdisciplinary and pedagogical-practical references. On this basis, further research questions for possible future projects will be put up for discussion.*

Key words: mathematical giftedness, developing expertise.

WILLIAM STERN

William Stern (1871-1938) worked as Director of Psychological Laboratory (1916-1919) in Hamburg and his work gave an essential impulse for founding the University of Hamburg one hundred years ago (https://www.uni-hamburg.de/en/uhh/profil/geschichte.html). At the University of Hamburg, he worked until 1933. Due to the Nazi-regime, he left Germany and continued his work as professor at the Duke University, USA. He paid particular intention to questions about prerequisites for professions and related to that, he was very interested in the diagnostic of abilities. In 1912, he proposed a new way of interpreting the results of intelligence tests with calculation the IQ. Up to now, this is one of the most popular concepts in psychology.

Stern was also very interested in giftedness or talents and many of his considerations are still very important. To refer to only a few of them, we mention the conceptual distinction between intelligence and talent. Stern underlined the existence of *general intelligence* and *giftedness in special subjects* (talents). From his point of view, the specificity of talent arises from a directionality of the entire personality towards the corresponding area. Thus, talents include not only special abilities, but also other personality traits such as interest and perseverance.

For Stern, giftedness does not necessarily have to be cognitive in nature. In addition, he formulated already in 1916 that giftedness not in any case leads to achievement. In that regard, he emphasized the importance of environmental factors for the development of abilities taking into account the interplay between genetic aspects, environmental factors and personality.

> "Giftedness as such provides no more than an opportunity to perform, it is an inevitable precondition, but it does not imply performance itself" (Stern, 1916, p. 110, translated from German original quote by the authors).

In the first half of the 20th century, Stern was already able to lay the foundations for a modern concept of talent or giftedness with his considerations. What has happened in Germany since then – especially with regard to the domain of mathematics?

In order to be able to contextualize selected research efforts, we will first take a look at the school systems in Germany. On this basis, essential research approaches and results will then be presented as important elements of a broad spectrum.

SCHOOL SYSTEMS AND PERSPECTIVES ON GIFTEDNESS IN GERMANY

Developments in the Federal Republic of Germany (western part of Germany)

The school system in the former FRG separated, and still does so, students after 4 or 6 years of schooling (10 or 12 years olds) depending on their academic achievements. Thus, there were different schools and differences in curricula for low, average and high performing students beside school for students with special needs. In parallel, some schools offered curricula for all levels of performance – here, we do not go into detail. To separate students and not to overlook hidden talents one of the first big studies to identify potentials of students started in 1965 (Heller, 2014). The aim was to match individual profiles of giftedness with the appropriate school or possibilities of fostering.

However, until the 1980s there was a widespread rejection of research projects on giftedness in cognitive domains or even fostering efforts. Based on the experience of the the Third Reich, many argued that the formation of elites in the German society should be avoided. Instead, the focus was on overcoming social inequalities by supporting students with low performance. This was motivated by the conviction that under appropriate conditions every child can learn everything (Bloom, 1976, as cited in Weinert, 2000, p.7).

Research about giftedness increased with the beginning of the 1980s. The impulses of congresses like the congress about gifted children headed by the psychologist *Wieczerkowski* in 1980 (“The highly gifted child: medical, psychological and pedagogical perspectives”; cf. Wieczerkowski & Wagner, 1981) and the *Sixth World Conference on Gifted and Talented Children* in Hamburg resulted in establishing a group of interested and engaged researchers at the University of Hamburg, consisting of math educators, mathematicians and psychologists. Together they developed concepts for identification and fostering mathematically gifted students at secondary school level – exchanging their ideas with a working group at Johns-Hopkins-University in Baltimore (USA). Headed by *Kießwetter* a still running program (“Hamburger Model”) was established. Later in 1985, the William-Stern-Society Hamburg was founded; a group that offers fostering programs for mathematically gifted students, counseling of parents and that supports research on giftedness. These activities made an important contribution to the gradually change of the social attitudes against giftedness in western Germany.

Developments in the German Democratic Republic (eastern part of Germany)

In the GDR, there was a uniform school system, i.e. a school for all, in which all students learned with the same textbooks and according to the same curricula. The aim was a high level of polytechnic general education and the development of the "socialistic personality".

In spite of the uniform school, there were no reservations against the promotion of special talents. On the contrary, the promotion of giftedness was seen as a social obligation (e.g. Dassow, 1983). From the beginning of the 1960s at the latest, this was especially true for mathematics,[1] which was regarded as fundamental to the entire field of natural science and technology because of its character as a structural science. Since the possibilities for promoting giftedness in the teaching of the uniform school were limited, a wide variety of extra-curricular activities was developed. This included systematically organised regional and supra-regional math clubs, correspondence circles, two math magazines for students ("alpha", "Die Wurzel")[2], an extensive book series for students in which nearly 150 volumes were published until 1990, and above all the Mathematics Olympiad as a state-wide multi-stage competition. This enjoyed very high media attention; especially in the first two stages, the participation rate was extremely high. However, this also led to the fact that the entire area of extra-curricular promotion of mathematics was very competition-oriented. In addition, in the GDR there were up to 14 special schools of mathematics, science and technology with their own curricula and final examinations.

Developments in reunited Germany

After reunification, the West German school system was initially adopted for the entire country. However, it was also possible to maintain a large number of special schools for mathematics and science in the eastern part of the reunified Germany.
In many regions, there have been more and more community or comprehensive schools for several years, in which almost all children learn together after the 4th grade. In addition, the special needs school system has been reduced for many areas, so that the heterogeneity of learning groups and thus the promotion of special talents (albeit in a reduced manner) has increasingly come to the fore in research and practice.

Fostering projects for mathematically gifted students were continued or newly developed at various university locations. The Mathematics Olympiad was established as a major nationwide competition. A large number of dissertations on mathematical giftedness were written in several university working groups. Currently, there is a nationwide project "Leistung macht Schule" (LEMAS), which is strongly funded by the state and aims to promote high-performing and potentially high-performing students in various school subjects, including mathematics. The project will support school development and networking in particular, and aims in strengthening teacher abilities in diagnosing and promoting giftedness amongst others in the STEM fields.

GERMAN RESEARCH IN THE DOMAIN OF MATHEMATICAL GIFTEDNESS

Due to reasons of space the following description of important research approaches and results offers (only) a selection of mathematical didactic work or psychological work with very strong mathematical didactic references, not taking into account research about more sociological aspects of giftedness like gender specific (e.g. Benölken, 2013) or classroom related research (Nolte & Pamperien, 2017).

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[1] In 1962, the GDR government passed a resolution for systematic development of extra-curricular offers for students in the field of mathematics ("mathematics resolution", in German: "Mathematikbeschluss").
[2] Cf. https://mathematikalpha.de/alpha and https://www.wurzel.org.

Nature of Mathematical Giftedness

In all models of mathematical giftedness we know, the core element is seen in a high level of the cognitive domain. Nevertheless, they have in common to take into account influencing factors like intrapersonal and environmental aspects and their interconnectedness. Thus, the developmental process is essential for shaping the potential. Furthermore, used characteristics of the cognitive domain can be related to different levels, which are illustrated in the following figure. While the level of thought and action patterns is purely descriptive, the levels of abilities and the underlying elementary mental processes and structures are explanatory.

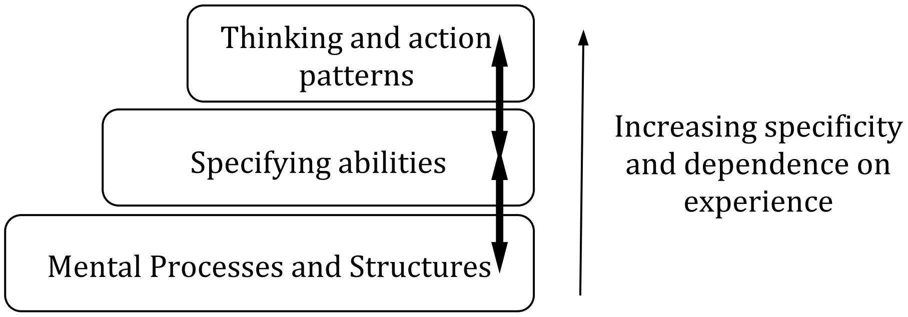

Fig. 1: Description levels for the cognitive core of mathematical giftedness (cf. Fritzlar, 2013)

We assume that specific or specified abilities develop over time based on more general mental processes and structures through mathematical activities. On appropriate occasions, these can then manifest themselves in characteristic patterns of thought and action.

In the GDR, psychological studies of mathematical activities and abilities were conducted relatively early (e.g. Gullasch, 1971). Building on this, the Academy of Pedagogical Sciences developed several qualification (PhD-) projects on mathematical giftedness and its promotion – later also from a more pedagogical and didactic perspective.

The dissertation by Dassow (1983) is an early empirical work on mathematical giftedness in primary school. He investigated a method for early identification of mathematically gifted students at the age of 9 to 10 years. For this purpose, he developed a theory-based model of the structure of mathematical giftedness and empirically examined it in a group of 60 students. The model includes the following abilities:

- Ability for mathematical abstraction, for rapid and comprehensive generalization of mathematical objects, relationships and operations
- Ability to think elastically
- Ability to curtail the mathematical thinking process
- Ability for logical and systematic sequences of thoughts
- Ability to quickly and easily change to reverse thinking

The first one is of particular importance; but overall, these abilities can only be found in elementary form in such young students.

Additionally, the model includes personality traits in the sense of action dispositions, which are regarded as necessary components of mathematical giftedness:

- Strong interest in mathematical activities
- High mental activity
- Strong self-confidence
- High degree of independence in mathematical activities
- Low fatigue in mathematical activities

The above listed abilities are partly based on elementary mental processes (lower level in Fig. 1). For example, switching to reverse thinking is understood as bidirectionally use of network connection, mental bonds sensu Krutetskii (1976). Therefore, Dassow was unable to develop test items that mainly test the extent of this ability (due to the occurance of these connections in almost all tasks); thus, it remains the only component of the developed structural model of mathematical giftedness that could not be empirically supported in the study. In addition, the theoretically derived special importance of interest in mathematical activities could not be confirmed.

At that time, there was no comparable scientific interest in the field of mathematical giftedness in the FRG. Only Kießwetter attempted to characterize activities of mathematically gifted secondary school students. While Dassow approached mathematical giftedness from a rather psychological point of view, Kießwetter followed a mathematical didactic perspective.

According to Kießwetter, theory-building processes are characteristic for the research work of mathematicians (Kießwetter, 2006, Fritzlar, 2008). In analogy, his approach for fostering gifted students is based on offering opportunities for age appropriate similar processes. At primary grade level, problems fields open an access for typical thinking processes like argumentation and generalization, at higher secondary level, the students even develop small mathematical theories.

Based on indicated characteristics of doing mathematics and the results of Krutetskii, at the beginning of the 1980s Kießwetter constructed a "catalog of categories of mathematical thinking" or "patterns of action", which are useful in working on mathematical problems. Using them in mathematically rich situations gives hints for a high mathematical potential. On that basis, Kießwetter developed the "Hamburg Mathematical Talent Test" (HTMB) whose problems can be solved by using these patterns: organizing material; recognizing patterns or rules; recognizing problems, finding (constructing) related problems; changing the representation of a problem and recognizing patterns and rules in this new area; comprehending very complex structures and working within these structures; reversing processes (cf. Wagner & Zimmermann, 1986). However, this catalogue is explicitly not intended to provide a comprehensive description mathematical giftedness (e.g. Kießwetter, 1985). It seems important, that it includes relatively complex patterns of mathematical operations, which, even more than Krutetskii's catalog of mathematical abilities, are formed by specific experiences and solidify and expand with them (see Fritzlar, 2010).

In Germany, the description of mathematical talents in primary school age developed by *Käpnick* had a great impact on the scientific discussion. Käpnick worked out theoretical and empirical mathematics-specific features of a potential mathematical talent, which he combined with general behavioral personality traits (Käpnick, 1998). This description was later incorporated into a talent and performance development model that is strongly oriented on the "Differentiated Model of Giftedness and Talent" by Gagné (2004).

Selected Characteristics of Mathematical Giftedness

If one examines more closely, what is meant with characteristics used for describing mathematical giftedness in the literature and how the authors try to capture their manifestations, then different accentuations become clear in some cases. In recent years, there have been increased efforts in Germany to elaborate such characteristics and possible relationships between them in more detail.

An outstanding example for these efforts is the dissertation of Aßmus (2017), which examines the cognitive characteristics of mathematical giftedness not only empirically for second grade students (8 years olds), but also profoundly theoretically.

One important result among others is the detailed theoretical analysis of the component "reversing thought processes". On this basis, she succeeds, unlike Dassow, in proving that the ability in reverse thinking is better developed in mathematically gifted second graders than in students of the same age who are not mathematically gifted. The advance is based above all on the ability to recognize reverse questions and situations. In contrast, the correct revising of a longer train of thought is a great challenge for all students of this age group.

For some characteristics of mathematical giftedness, a psychological perspective can be very fruitful. *Klix* describes four "basic components" of thinking which decisively determine the cognitive capacity of an individual: analogy, complexity reduction, multiple classification and multimodality or double representation. In this context, multimodality is understood as simultaneous use of several modality-specific representations and double representation is understood as simultaneous use of visual and symbolic representations in coping with cognitive demands (Klix, 1992, 1993).
The psychologists Krause, Seidel et al. (1999) investigated the double representation hypothesis for mathematical problems. As a result, they found that mathematically gifted students show better performance. They were able to solve more problems than comparison groups and solve them faster. The better performance, however, could not be explained by the traditional measures of experimental psychology; in terms of IQ, visualization or memory capacity; corresponding differences between the two groups of subjects were not significant. By means of EEG analyses, however, it could be shown that in mathematically gifted students, already within fractions of a second of understanding the instruction, those brain regions are activated which are responsible for the conceptual and pictorial-vivid modality. In the comparison group, however, such a double activation was not detectable. (Fritzlar & Heinrich, 2010).

These neuroscientific findings at the lower level in Fig. 1 support the indicators for mathematical giftedness at higher levels formulated by Kießwetter or Käpnick. Psychological investigations are also available, for example, for analogizing by mathematically gifted students (Foth & van der Meer, 2013). Detailed mathematics didactic studies in this area are currently carried out e. g. by Aßmus and Förster (2013).

Relations between giftedness and other theoretical constructs

Giftedness and IQ

It is often argued that the validity of the IQ for subject-specific requirements – for example problem solving in mathematics – is controversial and limited. Among other things, Nolte investigated this problem as part of a research and fostering project for elementary school children at the University of Hamburg, which celebrates its 20th anniversary this year.

Since there must be a selection in the city of Hamburg among the third-graders interested in participating in the fostering project, the children are initially invited to trial lessons – the "math club for math fans" (Nolte, 2004). Afterwards, the students work on a specially developed mathematics test and an intelligence test, which correlates particularly strongly with performance in mathematics. Important results from the study of more than 1660 students from nine years, of which complete data are available, can be summarized as follows (see Nolte, 2013a): The results of both tests correlated with -0.34. However, this correlation was significantly weaker for children who achieved particularly good results in the math test; for example, the correlation coefficient for positions 1-15 dropped to -0.14.

The not very strong statistical relationship and its further reduction are partly due to the (increasing) selectivity and the (decreasing) sample size. However, the results show that a special mathematical giftedness cannot be deduced from the IQ (Nolte, 2013). The doubts expressed by mathematical didactics on a simple connection between IQ and giftedness (e.g. Bardy, 2007, Bauersfeld, 2003, Käpnick, 1998) are thus further supported.

Mathematical giftedness and developing mathematical expertise

Originally, 'giftedness' and 'expertise' were understood as two fundamentally different constructs that have their roots in different research traditions. While 'giftedness' refers to an area-specific potential of high performance, 'expertise' is characterised by continuously outstanding performance levels in a certain domain. However, nowadays it is undisputed that giftedness does not 'evolve' on its own. Rather, it is based on long term learning processes and practical experience which themselves are reliant on support and stimulating encouragement functioning as inter- and intrapersonal catalysts.

Therefore, there are currently increased efforts to synthesise both approaches, resulting in integrated models, which explain the development of giftedness, talent or expertise. For the domain of mathematics, Fritzlar (2015) described a model of developing mathematical expertise. For reasons of space, it is not possible to go into this model in detail in this article. However, some of the consequences, which we believe to be essential, are listed below:

- Individual factors, such as the cognitive apparatus or its genetically predetermined part, will lose their anticipatory nature in the model of developing expertise. This is true also for the current (measured) level of expertise, which does not allow any reliable statements on which level will or can potentially be reached (Sternberg, 1998).

- Rather, the model emphasises that any individual must continuously progress in his or her development through a suitable interplay of all participating factors. This way, he or she has the chance to reach the level of expertise that enables access to fostering programmes etc. and/or lead him to be identified as 'gifted' (Sternberg, 2000).
- Developing extraordinary performance is generally not understood as an autocatalytic process. The environment, in this context, does not merely serve in a defensive function by preventing potential 'disturbances'. Rather, experience in mathematics and thus opportunities to learn are necessary preconditions to develop one's expertise. This also points to the responsibilities of schools and society in supporting necessary long-term and specific learning activities.
- From a pedagogical perspective, inclusive fostering activities should at first be given priority over exclusive ones.
- A (amongst others) cognitive 'basic configuration' is differentiated from mathematical abilities and special achievements. This points to and considers their context-dependence and therewith, for instance, the problem of identifying gifted students or initiating positive feedbacks.
- The model brings to the fore the systemic character of expertise (cf. Ziegler & Phillipson, 2012): the current level of expertise appears simultaneously as an emergent feature and as an element of a system that is characterised by cross-linkages, dynamics and equifinality, amongst others.

Looking Ahead

In the past decades, there was a large number of research projects in the field of mathematical giftedness in Germany. Nevertheless, from our point of view, there are still many open questions, the answers to which could be worthwhile. On the one hand, it seems important to us to continue theoretical work in the form of a more detailed elaboration of possible characteristics of mathematical giftedness, relationships between these and relationships between giftedness and other constructs. On the other hand, the practice-oriented further development of promotion approaches is important, which, for example, also include children with special support needs. In this respect, we consider the concretization of a systemic view of mathematical giftedness and its promotion to be very promising.

Theoretical developments

The construction and use of structures (or abstraction and generalization) seem to be of central importance for mathematical giftedness. So far, however, this characteristic has been investigated in German-language studies predominantly in connection with geometric or number patterns. Already in Krutetskii's classical book, the *ability for formalized perception of mathematical material, for grasping the formal structure of a problem* is seen as an essential component of mathematical giftedness. However, this refers to a more general concept of "structure", in which a structure is understood as a set of elements that are in (different) relationships to each other; the set with its internal relationships then appears as a whole, underlying the corresponding mathematical situation. On this basis, it seems promising to us to take a closer empirical look at the ability to grasp underlying mathematical structures, structural similarities and differences between math-related situations and to investigate its development across different age groups.

An important related question refers to change which arise with focusing on investigations in regular math lessons. A first study with primary school students suggests that more students see mathematical patterns than 15 years ago. Therefore, it seems questionable to us whether seeing patterns can still be regarded as a hint for giftedness (Nolte & Richter 2019).

Those who work with mathematically gifted children and adolescents have certainly often experienced a problem solver suddenly seeing the solution. Already Krutetskii (1969) describes these observations impressively and, above all, attributes them to a very rapid succession of solution steps: "... that in several cases one gets the impression that, in essence, there is no process, but rather that there is an analytico-synthetic "vision" of the mathematical material in a single act, a single step"(Krutetskii, 1969, p. 108). This, in our view, can be related to the somewhat broader construct of *intuition* introduced by Käpnick (2012) into German-language research on mathematic giftedness. This construct also points to the importance of (possibly implicit) knowledge. In terms of research methodology, the phenomenon of intuition (with possible references to knowledge and speed of information processing) is certainly very difficult to access, yet it seems highly interesting and worth further empirical efforts.

Creativity and giftedness are often seen in close connection. However, there is no agreement on the exact relationship between the two constructs. Thus, creativity is sometimes seen as a prerequisite, as a possible component or a possible consequence of giftedness, or both constructs are seen as independently of each other. A clarification of the relationship is difficult also because both constructs are partly fuzzy. Assmus and Fritzlar (2018) make a proposal that combines a modern concept of giftedness with a relativistic understanding of creativity. Here, further scientific efforts seem important to us, with which in particular the question could be investigated how creative mathematical activity at (primary) school age can be expressed itself and how this can be promoted.

Practice-oriented developments

Discussions about the realization of inclusion also underline the necessity to foster students with a high potential. Nolte's study about twice exceptional students (Nolte, 2013b, 2017) underline the importance of interdisciplinary collaboration. The special needs of these students do not belong to the usual professional knowledge of mathematics teachers. Yet, not enough is known about the special needs of mathematical gifted and promising students who must overcome or handle barriers in their learning processes.

Ziegler (2005) used the Actiotope Model to develop a systemic view of giftedness that comprises four components: the individual's action repertoire, its subjective action space, its goals, and the environment surrounding the individual. In our view, the next step should be to attempt to specify this model for the domain of mathematics and to derive from it possible conditions of success for the promotion of mathematical giftedness.

Going back to Kießwetter's patterns of action we question how it is possible to use the results of research in the field of giftedness for working with all students. Nolte (2006) distinguishes between patterns of action which can be observed and "action" from cognitive components of problem solving which may lead to an action but are not observable. Thus, for instance. students can be taught how to organize material as an observable action. But, we cannot lead a student to recognize pattern and structures. Thus, inclusive fostering activities in regular classroom may be based on explicitly discussing patterns of action and cognitive components. Even though there has been more or less research on giftedness in Germany for more than 100 years, much remains to be done, at least with regard to the domain of mathematics.

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TEACHING HIGHLY ABLE LEARNERS IN DIVERSE CLASSROOMS: PEDAGOGICAL POSSIBILITIES THROUGH COLLABORATION

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We argue in this paper that, with appropriate support, teaching of highly able learner can occur in diverse classrooms. We draw on a constructivist theory of learning and a differentiation paradigm (Dai & Chen, 2013). The claim that teachers can orchestrate teaching for highly able students in diverse classrooms is considered with evidence of our own and other data, warrant, backing, qualifier and rebuttal. Results from many studies have given knowledge of learning needs of mathematically highly able learners as well as of successful teaching to meet a diversity of learners. Drawing on our research, and work with school development, we share ideas about possibilities for teachers to support learning for all students, that is, including the highly able, within a diverse classroom. In particular, we advocate the possibilities from professional collaboration and our practice examples illustrate this claim.

Keywords: Gifted education, Differentiated instruction, collaboration

INTRODUCTION

The importance of improving student results and raising the status of the teaching profession are often highlighted in government policy and media. Many countries across the globe have looked for ways to improve their education systems and international rankings of student achievement. However, there are several studies reporting that highly able learners are not given the educational support they need (e.g., Leikin & Stanger, 2011; Pettersson; 2011). The first author of this publication personally asked Françoys Gagné and Linda Silverman, both leading researchers in the field of gifted education, if they believe it is possible to teach highly able learners in diverse classrooms. Both quickly responded: No. This is a contradiction to our experience, research and beliefs as educational researchers and teachers and in this paper we propose an alternative pedagogical possibility.

It is reasonable to assume that differences in perceptions of what is possible or not, partly has its roots in which definition, or paradigm, of gifted education is used. As researchers in pedagogical work we acknowledge that students have differing inner qualities; some have higher cognitive abilities, others have lower cognitive abilities. There are also differences in terms of learning dispositions, motivation, volition and wider individual experience. Consequently, we acknowledge that some students have possibility to excel faster and deeper into a subject, for example mathematics, than most other students. Aiming to support students to develop as far as possible in knowledge, the process of how to recognize and support students' learning should be in focus. Moreover, since most highly able students are placed in diverse classrooms (Shayshon, Gal, Tesler, & Ko, 2014) it is important that developmental research includes focus on teachers' practice.

Through our research and experiences from school development, Erasmus and other projects, this paper presents an argument that it is possible to teach highly able learners in diverse classrooms.

THEORETICAL CONSIDERATIONS

The aim of this paper is to give an argument on that highly able students can be taught according to their learning needs in diverse classrooms. The theoretical perspective of learning is constructivist, and the chain of argument is inspired by Toulmins' (1958) model of argumentations as described by Brunström (2015, p. 19[3]). The model is described by six parts that are connected. It was developed to analyze argumentations made in every-day life.

Claim: The statement that is going to be defended or the hypothesis or conclusion that has been drawn,

Data: Facts that the statement/hypothesis/conclusion is based on,

Warrant: Explanation to why the step from facts to statement/hypothesis/conclusion is reasonable,

Backing: Sometimes the *warrant* need support. Support giving the *warrant* validity is called "backing",

Qualifier: Marking the validity of the reasoning, that is how valid the step from facts to statement/hypothesis/conclusion is,

Rebuttal: In some circumstances there may be exceptions from where the conclusions are not valid. Such circumstances are called "rebuttal".

Constructivism is a theory of learning which advocates the acquisition of knowledge as an individually tailored process of construction. To design effective teaching environments, constructivism promotes understanding of what children already know when they come into the classroom. Curriculum should be designed to build on students' experience and knowledge and allowed to develop with them. The teacher acts as a facilitator who encourages students to discover principles for themselves, to construct knowledge through open-ended discovery and problem-solving. To do this, a teacher should encourage curiosity and discussion amongst students as well as promoting their autonomy (Ernest,1995; Steffe & Gale, 1995).

LITERATURE REVIEW

In this section we elaborate on different views of gifted education based on the review by Dai and Chen (2013). Thereafter we continue with a brief overview of highly able students in school from a pedagogical perspective, focusing on their learning needs as characteristics to recognize them. Further we briefly elaborate on teacher professional development, collaborative work between teachers and principals' support to teachers to enable teaching inclusive of highly able students.

Three views of gifted education

Dai and Chen (2013) have done a thorough overview of different ways of addressing gifted education, as a result they divide the views into three paradigms: The gifted child paradigm, The talent development paradigm and The differentiation paradigm. Different

[3] Brunström (2015) is here translated to English.

answers will be given to the four questions; *What is high ability?*, *Why should school practice bother?*, *Who are the highly able?* and *How should teaching be adapted for the highly able students?* Responses to these questions depending on which paradigm is used.

The gifted education paradigms proposed by Dai and Chen (2013) are here very shortly summarized. In *The gifted child paradigm*, giftedness is something individuals are born with, school should adapt teaching for these students since they are the leaders of tomorrow, high IQ is a frequent example of how a student may be defined as highly able or not, and these students need special schools with a designed curriculum. In *The talent development paradigm*, high ability is something that can develop - although some individuals have potential to develop faster and deeper than others - the highly able students are seen to be the future leaders and the innovators that can solve the big problems of tomorrow. IQ measurements are not enough to find who is highly able or not – other standardized and also non-standardized measurements are of importance, and these students need special school or special classes where acceleration and enrichment are possible ways to adapt teaching. In *The differentiation paradigm* high ability is something that can be developed - and all individuals are seen to have capacity to develop - the school is important to support all students to develop as far as individually possible - the highly able students can develop far more than expected according to age, teaching should be adapted so that all students are given possibilities for learning.

This paper addresses the diverse classroom, therefore *The differentiation paradigm* is the only possible view. We do however acknowledge that students have differences and that these depends on both genetics and environment. That is, we acknowledge the two other paradigms answers given to *Who is highly able?*. But our focus is on the question *How should teaching be adapted for the highly able students?*

Highly able students in school

The reason to identify students learning needs, from a pedagogical perspective, is to be able to adapt teaching towards their learning needs. That is, to offer teaching that enables learning so students can develop as far as possible in knowledge.

In this paper highly able students in mathematics are seen as a sub-domain to highly able students in general; they are assumed to share the same learning needs. That is, they are quick learners, prefer complex and abstract tasks (Leikin, Leikin, Paz-Baruch Waisman, &, Lev, 2017; Rogers, 2007; Tomlinson, 2016), can get bored and even drop-out of school when teaching goes too slow (Mohokare & Mholo, 2017). This paper address high ability in general, with students with high ability in mathematics as a specific cohort example.

To successfully teach in diverse classrooms, Le Fevre, Timperley and Ell (2016) suggest that teachers need to have knowledge of their students' learning needs and to be able to adapt teaching towards these needs. This means that to be able to teach highly able students in diverse classrooms teachers need to be able to first recognize the students' learning needs and second orchestrate teaching to include those students (Mattsson, 2013).

Most highly able students do not need repetitions (e.g., Persson, 2010), their learning may even be hindered if they are forced to it (Rogers, 2007). They are fast learners, in mathematics for example they need less time than other students to solve complex tasks (Nolte & Pamperien, 2017). Many studies report that acceleration is suitable for highly able students (e.g., Colangelo & Assouline, 2009). However, Sheffield (2015) means that acceleration may be contra productive if the aim is to increase the number of individuals in mathematics intensive occupations.

In general, students with high ability in mathematics easily grasp the formal structure of a problem and quickly generalize (Krutetskii, 1976; Sheffield, 2003). They may show great interest on why and how a solution to a problem gets right or wrong (Mohokare & Mhlolo, 2017; Sheffield, 2003). It does not necessary mean that highly able students become successful in achievement, they may become bored and act rebellious for having to work on a level they perceive too low for them (Mohokare & Mhlolo, 2017).

To be able to meet highly able students' learning needs, several studies have concluded that teachers need professional development on gifted education (e.g., Shayshon et al., 2014). According to Desimone (2009) collaboration is one out of five core features making teachers' professional development successful, that is to improve practice and increase teacher knowledge and skills. Collaboration between colleagues is by teachers perceived to be necessary to orchestrate teaching meeting the learning needs of highly able students in diverse classrooms (Mellroth, 2019). Collaboration with students is also an integral element of pedagogy and recommended between teachers and parents of highly able students (Porter, 2008). However, this paper focuses on the collaboration between teaching colleagues.

In Sweden, the principal is the pedagogical leader on her or his school. S/he is responsible that each student receives an education that develops their learning. For principals to support such education, Forssten Seiser (2017) found that they should focus on teachers' prerequisites to develop students' learning environments. They should encourage teachers to try different ways to adjust their teaching, they should also strive to create a common understanding of what is needed, from the school, to give students learning support. Related to this paper, the findings of Forssten Seiser (2017) is interpreted as meaning the principal has an important role in supporting and encouraging teachers to learn about highly able students' learning needs and gifted education. In the study of Johnsen, Haensly, Ryser and Ford (2002) principals showed interest and were involved in a professional development project on gifted education, and encouraged the participating teachers. In the end of that project the majority of participating teachers had positively changed their classroom practices and adapted teaching to include highly able students in teaching.

CLAIM: THROUGH COLLABORATION, TEACHERS CAN ORCHESTRATE TEACHING HIGHLY ABLE STUDENTS IN DIVERSE CLASSROOMS

One claim on teaching highly able students in diverse classrooms will be given here. In this paper it is that, through collaboration, teachers can orchestrate teaching for highly able students in diverse classrooms. The example follows Toulmins' (1958) model of argumentation presented earlier. The claim is considered with evidence of our own and other data, warrant, backing, qualifier and rebuttal – following the chain of argument is inspired by Toulmins' (1958) model of argumentations noted earlier in this paper.

Data

Mellroth (2019) showed that teachers who participated in a two-year long professional development program on gifted education perceived they had competence in recognizing highly able students in diverse classrooms. Those teachers connected their teaching practice with theories of gifted education, for example they showed knowledge, in line with research, of highly able student characteristics, in general, as well as in mathematics.

They were aware of that their own mathematical knowledge may be to low to meet the learning needs of mathematically highly able students. Low mathematical knowledge among teachers is shown to be a problem when teaching mathematically highly able students (e.g., Hoth et al., 2017). However, the teachers in the study of Mellroth (2019) gave a solution on to how to solve the problem. They acknowledged that teachers and other school staff create a team of colleagues. Therefore, they suggested that teachers with deeper mathematical knowledge can be found through collaboration with colleagues. Those teachers can give support to teachers with less mathematical knowledge on how to orchestrate teaching for mathematically highly able students.

Earlier work by Margrain (2005, 2010, 2011) explored the experience of highly able young children in early childhood and with transition to starting school. Through classroom observations, teacher and parent interview, and document analysis, evidence was gathered of how teachers worked in various ways to understand diversity, differentiate curriculum, and document learning. The teachers worked particularly closely with parents and colleagues to gain understanding of the competencies children displayed in varying contexts, and to share responsibility for supporting development. A way of documenting such collaboration is through the Individual Education Plan (IEP), advocated by Mazza-Davies (2008) and used in Margrain's own school teaching experience.

Warrant

By training teachers (n=74) in differentiating instructions, over a period of two years, to meet the learning needs of highly able students, Johnsen et al. (2002) showed that, by the end of the second year of the program, none of the participating teachers continued to ask their students to wait while other students finished, 57% of the teachers used pre-assessments to compact the curriculum, provide acceleration or enrichment, 77% of the mathematics teachers chose to accelerate instructions and 71% of the teachers began offering a variety of learning activities, compared to 13% before training. The project, seen as a professional development program, supported 99% of the participating teachers to change their classroom practices to adapt for highly able students. Collaboration between teachers and other project members is highlighted as important for the developed internal and an external support structure in the project. Therefore, the study of Johnsen et al. gives a warrant to the subsequent data collected by Mellroth (2019) and Margrain (2005, 2010, 2011).

Backing

Nolte and Pamperien (2017) showed that challenging tasks, developed to support mathematically highly able students, were successfully applied in diverse classrooms, but that highly able students needed less time and achieved better regarding generalization and proving. To be able to use such challenging tasks in teaching, Nolte and Pamperien highlight that the teachers needed training on how to work with highly able students. In line with Mason and Johnston-Wilder (2006), they mean that teachers should ask questions to students which supports them become more independent in their solution process rather than giving them an answer on a problem. The study of Nolte and Pamperien gives back-up to the warrant based on the study of Johnsen et al. (2002). For example, they show that same kind of tasks can be used successfully for all students in diverse classrooms, but that teachers who orchestrate teaching need extra training to be able to implement them. That is, collaboration between teachers with deeper mathematical knowledge and knowledge of gifted education and teachers less skilled can be a solution.

A survey conducted by Margrain, Lee and Farquhar (2013) confirmed – unsurprisingly - that teachers who had experience working with high ability students were stronger advocates of these students and felt more confident in their practice. This study supports the argument that working with high ability students in diverse classrooms disseminates awareness of this group and facilitates skill in differentiation across the teaching profession.

Qualifier

Examples from three countries are used as qualifiers. First we share three systems-level examples of collaborative pedagogical support structures, followed by brief classroom-level examples of collaborative pedagogy which supports mathematically high ability students. Our experiences – including from the first author's work in an Erasmus Plus project, undertaken between 2017 and 2019 (European Comission, 2019) have illustrated examples of how teachers can be supported to recognize and orchestrate teaching inclusive of highly able students.

In The Netherlands we were told that each school ought to have a gifted coordinator. Several solutions are used in the country to give teachers in regular schools support on how to recognize and orchestrate teaching for highly able students. During the Erasmus project (European Comission, 2019) we visited and learned from an organization, DeDNKRS[4], where special education teachers, also trained in gifted education, gave support to and trained teachers in regular classrooms. In practice, the pedagogues from DeDNKRS were called in to counsel regular teachers at schools who needed guidance in how to support highly able students learning. During the time of counseling, regular teachers and pedagogues at DeDNKRS acted as colleagues, sharing knowledge of students and expertise in gifted education to improve practice.

In the federal state Hamburg, Germany, a political decision from 2014 (Landesinstitut für Lehrerbildung und Schulentwicklung, 2014) declared that each school should have a gifted coordinator. Through the Erasmus project (European Comission, 2019) we learned that this person aims to give support to teachers at the school on recognizing and orchestrating teaching for highly able students. To organize and guarantee the quality of these coordinators a central organization, Beratungstelle besondere Begabungen[5], (BbB), gives professional development for teachers and they administer networks between the gifted coordinators at schools in the regions of Hamburg. In addition, BbB works closely with the University of Hamburg which validates their scientific connection.

Thirdly, Gifted Aotearoa[6] is a New Zealand national network of expertise, established to improve quality of education provided to learners, and provides support through sharing professional expertise, growing local networks, nurturing local leadership and developing professional pathways. The five culturally-framed collaborative initiatives are: keeping connected, keeping afloat, learning together, leading together, and raising

[4] Link to the website of DeDNKRS; https://dednkrs.nl/
[5] Link to the website of BbB; https://li.hamburg.de/bbb/
[6] Link to the website of Gifted Aotearoa; https://www.giftedaotearoa.nz/

new leaders. Gifted Aotearoa as an organization is in itself a professional collaboration between professional learning providers and a parent advocacy organization, and the work is funded by the New Zealand Ministry of Education.

These examples from show that high ability is acknowledged in differing school systems. They can also be compared to the professional development program on which Johnsen et al., (2002) made their study on, but in contrast the examples from The Netherlands, Hamburg and New Zealand are not time-limited, they are part of the countries' school practices. Further, the examples show that examples of organizational structures give teachers support to recognize students who are highly able. Support by principals is an aspect shown to be of important to change practice by Johnsen et al. (2002) and Forssten Seiser (2017). We now give some brief examples of classroom-level practice from our own research to illustrate pedagogical possibilities.

In Mellroth, van Bommel and Liljekvist (2019) an example is given on how collaboration among teacher colleagues in diverse classrooms is perceived as a possibility to orchestrate teaching inclusive of highly able students in mathematics. The example comes from a teacher who on regular basis had another teacher in her mathematics lessons, a common way to arrange teaching in Sweden and in all subjects, called "the two-teaching system". The teacher declared that both were happy for this collaboration and recognized that through this arrangement they could work together to meet the learning needs of the highly able students. The main teacher contributed deeper knowledge of the individual class members which the other teacher needed to learn, and the other teacher had deeper mathematical knowledge and he therefore orchestrated challenges for the highly able. Thus, both teachers contributed essential elements to the collaboration for the benefit of students, and the two teachers were also able to learn from each other.

Margain's (2005) doctoral thesis documented several ways in which teachers could collaborate within a school to support curriculum differentiation. In one example, a five-year old student who was highly able in mathematics and reading had a 'home' school class with age peers, but attended classes with students several years older in the areas of strength. In another example an early childhood teacher met with teachers from a local primary school to borrow resources and share teaching ideas so that a four-year old could be given extension in the preschool setting. In a third example, the principal of a five-year old met with parents and teachers and an IEP was developed which attended to social, emotional and academic goals. In this final example no members of the school staff felt they had particular expertise in gifted education, but they recognized the high ability of the given student and were committed to student well-being and learning more as a professional, collaborative team.

For the claim given, these qualifiers validate that collegial work can make it possible for teachers to recognize and orchestrate teaching of highly able students in diverse classrooms. Important colleagues to teachers are in these cases gifted coordinators. These collaborations work in a similar way as special education teachers give support to teachers in meeting the learning needs of students with learning disability.

Rebuttal

The claim is based on two important conditions; Firstly, teachers who can orchestrate teaching inclusive for highly able students have professional development in gifted education. Secondly, the organization has a structure for teachers to collaborate with other colleagues. If these two conditions are not fulfilled there is a rebuttal to the claim.

To be able to recognize highly able students, teachers need to have knowledge of their characteristics and also opportunity in their practice to apply this knowledge. That is, they should have undertaken some sort of professional development on high ability either in their teacher education or through professional development for in-service teachers, and they must have the support of school leadership in their work with high ability students. The teachers in the studies by Johnsen et al. (2002), Mellroth (2019), and the staff at BbB and at DeDNKRS all had some kind of professional development in gifted education. In addition, Mellroth (2018) indicates that collaboration and discussions among teacher colleagues are also of importance during professional development, which according to Desimone (2009) is a core feature for successful professional development.

Another rebuttal is that mathematically highly able students cannot be taught only in the diverse classroom, they need to sometimes work together with like-minded peers (Nolte & Pamperien, 2017; Rogers, 2007; Szabo, 2017). This means that they on regular bases, perhaps one time every week, need to work with challenging mathematical tasks in groups of mathematically highly able students. Providing this extension opportunity does not negate the importance of differentiated instruction within the regular classroom.

LIMITATIONS

This paper gives only one claim to why highly able students can be taught in diverse classrooms, the data and given arguments comes from a relatively small number of studies. Further, no counter arguments are given. Naturally further work is needed to strengthen the argument. We are aware that the systems-level examples of professional learning and collegial support are not universal or generalizable.

The claim with its argumentations gives an example that it can be possible for teachers to recognize and support highly able students in diverse classrooms. To be able to change practices to better meet highly able learning needs, positive examples can be used as role-models and encourage principals and teachers to adapt teaching inclusive also for highly able students. To achieve this outcome, systems-level organization and support needs to occur, including policy level commitment to gifted education, professional learning, and time for collaboration.

ETHICAL CONSIDERATIONS

Our own studies have been approved by university ethics processes and there are no ethical concerns for this paper. Our work addresses the ethical issue of *beneficence* through advocating for the needs of students and teachers.

DISCUSSION

We do not suggest that well acknowledged and experienced researchers like Gagné and Silverman are wrong when they say it is impossible to teach highly able learners in diverse classrooms. One reason to our different beliefs can be that we differ in what we mean by ‘possible to teach highly able learners in diverse classrooms’. Gagné and

Silverman are both educational psychologists while we are pedagogues. Maybe a difference is that they as psychologists focus on individual's inner life, while we as pedagogues focus on students' learning. We agree on that highly able students probably will feel different than most other students in the diverse classroom. But we think it is possible, although not easy, to orchestrate teaching in diverse classroom that give possibilities for highly able learners to develop in knowledge.

Pedagogical support for students with high abilities in diverse classrooms is enhanced through the collaborative practices. The examples given in this paper point to: a constructivist theory and philosophy of education which recognizes diverse individual learning; systems-level support systems which develop collaborative opportunities between teachers and sharing of expertise; school leadership support; professional learning opportunities; and curriculum differentiation. Operationalisation of these elements requires time, money and commitment – justifiable on the basis of student and teacher benefit. Practice and research should be intertwined to support and strengthen each other aiming develop both practice and research. Our research enables us to review the questions by Dai and Chen (2013). *What is high ability, and who are the highly able* is framed by constructivist theory and recognition of individual diversity in learning. The paper has presented a range of examples to illustrate how teaching of high ability pupils can occur in the context of diverse classrooms as support for the question, *How should teaching be adapted for the highly able students?* In particular, professional collaboration has been claimed as an example of how this can occur. Finally, to respond to the question, *Why should school practice bother?,* this paper has presented arguments for the claim that it is possible to teach highly able students in diverse classrooms. This finding is critical to share because, pragmatically, it is the only solution in many contexts. Firstly, the philosophies and funding of education systems in many countries means that separate gifted education systems are unlikely to be offered. Second, the needs of students in rural and remote regions must be considered, where there is less likelihood of a viable cluster of highly able students. Thirdly, highly able students are not a homogenous group, even within a particular subject area such as mathematics. Teachers must always be alert to individual differences in learning style and pace of learning, this is the essential work of contemporary pedagogues and essential for student support. Building on collaboration yield promise for teachers to engage in this critical work.

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UNCERTAINTY AS A CATALYST FOR MATHEMATICAL CREATIVITY

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Abstract. *In mathematics, uncertainty in the form of constraints takes on different forms. Constraints in problems in different domains within mathematics have played a unique role in its development. Paradoxically, constraints are both restrictive as well as liberating. For instance a result that is very generally stated and difficult to prove can be tackled by imposing constraints on the premises, i.e., restricting its scope. Hardy and Wright once noted that the proof or disproof of the twin-prime conjecture" is at present beyond the resources of mathematics." It remains unproven even today. On the other hand, when an impasse is reached on a problem, it is sometimes overcome with the insight of inventing new tools to go beyond the (perceived) constraints of the problem. In this process of liberating the problem from the shackles of uncertainty, a different form of Uncertainty arises in the acceptance/un-acceptance of these "new tools" or "new methods" by the field. Both mathematicians (in the history of the subject) and students of mathematics have experienced the frustration when the field or the teacher respectively, do not accept the methods used to overcome the constraints. This dynamic view of insight overcoming constraints and/or insight imposing constraints is also ubiquitous with many heuristic devices used by mathematicians to solve problems. In this plenary, we will survey the role of Uncertainty in the dynamic of insight/constraints as catalysts for creativity. Relevant theories of creativity are used to scaffold the contents of the lecture.*

Key words: Arithmetic; "Big C" creativity; Continued Fractions; Euler; Infinite Series; Lambert; Logarithm; Mathematical creativity; Napier; Uncertainty

INTRODUCTION

Creativity is a word that has increasingly become ubiquitous with 21st century life- a "sign of the times" so to speak. It is a word used in abundance. The uses range from the development and use of apps in the "gig" economy, advances/solutions offered by management agencies for the IT sector and health sector, brochures outlining "goals and outcomes" printed by schools and higher educational institutions, and proposals for alleviating local and global problems caused by conflict induced mass-migrations or climate change. This is not an exhaustive list by any means. In this paper, the term creativity is not used in an everyday or tautological sense but in a more domain specific sense. Mathematical creativity is used to refer to advances made by particular individuals to the development of the field of mathematics. It includes both the criteria of novelty (originality) as well as usefulness. In other words, it refers to what is known as "Big C" creativity, i.e., creative acts that involve extraordinary individuals carrying out extraordinary thought processes (Sternberg, 1999, p.148). There is a personal dimension to the creativity described in this paper, as the acts were "new and useful" for the person doing the creating in addition to pushing the known boundaries of mathematics in that time period. Over time these acts have also gained consensus as "Big C" creativity by the community of mathematicians, hence they can also be considered to meet the burden of creativity within the social/professional context of mathematics.

THEORETICAL FRAMEWORK

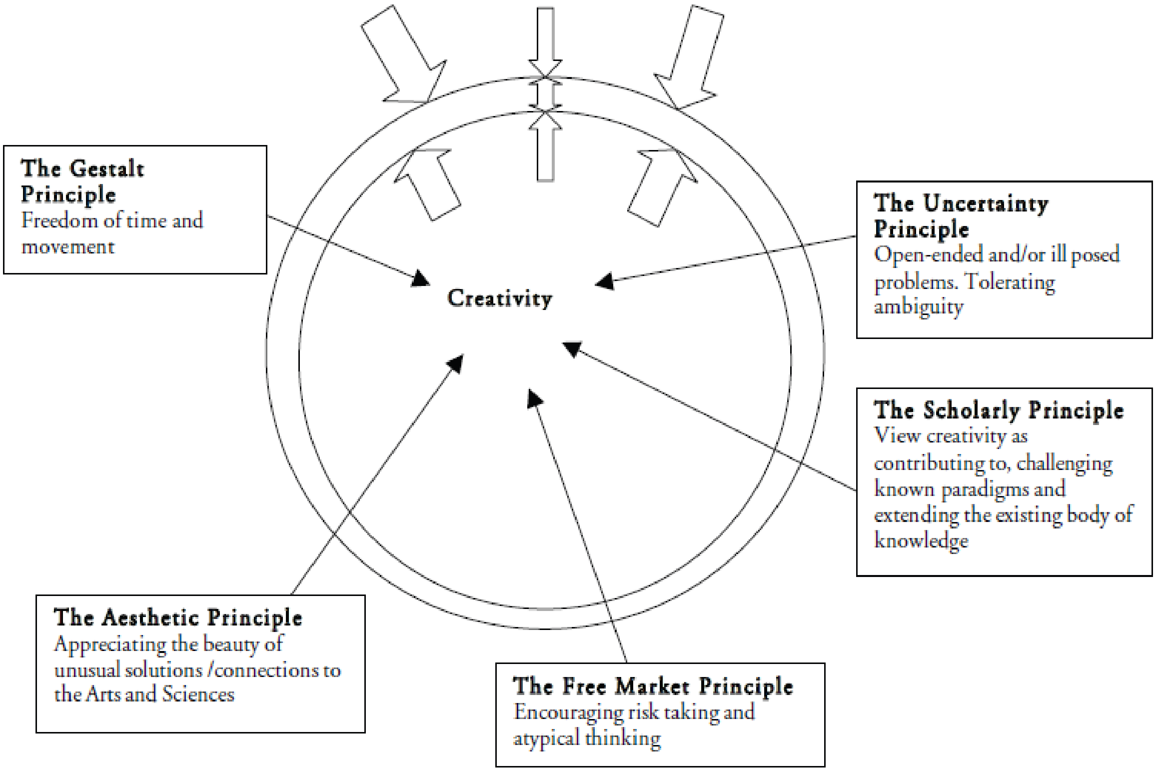

Figure1. 5 Principles to maximize creativity (Sriraman, 2005)

The theoretical framework used is constructed out of Sriraman's (2005) model of creativity which includes 5 principles to maximize creativity in the classroom distilled from the existing literature on creativity. These principles were distilled by studying conditions that were conducive for creativity to occur at the professional level and theorizing the possibilities for classrooms. Haavold (2016) empirically validated this model by operationalizing the 5 principles in an instrument administered to high school students in Norway and found that the principles were robust. Further he found achievement and motivation were two predictors of mathematical creativity in that particular context. In a different study with university students in the U.S enrolled in a proofs course, Regier & Savic (under review) found that the use of the five principles by the instructor contributed to greater self-efficacy in the students for proving tasks.

While achievement is often conflated and misconstrued as performance in mathematics courses, it can also be viewed as the amount of preparation a person has undergone over time in their professional life as a mathematician when they engage with a problem or a new area of mathematics. Motivation is a broad affective term under which the notions of uncertainty and risk-taking are best described. Creating, as opposed to learning, exposes one to the uncertainty as well as the difficulty of inventing original ideas in mathematics. In the classroom setting, it requires the teacher to provide affective support to students who experience frustration at being unable to solve a difficult problem. Regier & Savic (under review) validate this in their study where the combined use of the principles contributed to more opportunities for students to build their self-efficacy.

Exposure to specific mathematical ideas or the evolution/solution of a particular problem from the history of mathematics that have taken the efforts of generations of mathematicians reveals the interplay of uncertainty with motivation in a broader social context. One also gets a glimpse into how constraints come into play in the field in the development of a mathematical idea. Zaslavsky (2005) reflected on the pedagogical uses and the meaning of uncertainty in her longitudinal work with different populations in Israel. She reported that tasks that contained elements of uncertainty were the ones that evoked positive affective dispositions in spite of the cognitive conflict. The tasks used in her studies were classified as (a) problems with competing claims (p.300), (b) open-ended problems with an unknown path or questionable conclusion (p.302) and (c) combinatorial and probability problems with non-readily verifiable outcomes (p.304). In this paper the notion of uncertainty that is explored includes elements of all three components expounded by Zaslavsky in a pedagogical context. However the focus is on the grey area of mathematical research in a professional context where new methods/techniques are needed in the face of uncertainty to solve long standing problems.

Beghetto and Corazza (2019) suggest inconclusiveness and uncertainty are important ingredients in any dynamic conceptualization of creativity. They criticize traditional views "tend to privilege static creative achievement and fixed creative traits, rather than focus on the more dynamic, developing, and variable nature of creative thought and action" (p.2). In the remainder of this paper several episodes from the development of mathematics are used to describe catalysts of mathematical creativity. Mathematical creativity is catalysed by five factors:

1. Knowledge of what is already known (labelled **Prolonged Preparation**)
2. Questions encountered at the boundaries of what is known that are unanswerable due to domain limitations with existing tools/techniques (labelled **Constraints**)
3. Uncertainty in what is possible with the questions that spur the development of something new, or what is not possible that imposes constraints on the development of something new (labelled **Uncertainty**).

Here an example is warranted since this is the main focus of the paper. Any new result or a new technique used by the field is characterized by a period of blind innovation. Many results and inductive techniques appear that at times seem "bizarre" to the untrained eye. The first example comes from John Wallis' exemplary work *Arithmetica Infinitorum*, one of the foundational texts for Newton, where the act of adding sums (before we had the general power rule for integrals) seems fantastical.

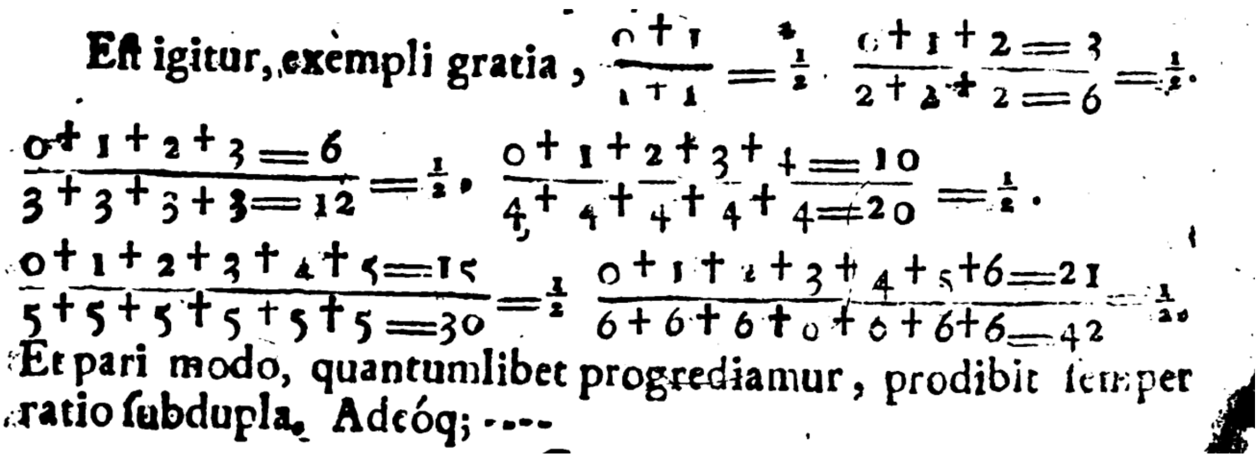
Eft igitur, exempli gratia, $\frac{0+1}{1+1} = \frac{1}{2}$. $\frac{0+1+2=3}{2+2+2=6} = \frac{1}{2}$. $\frac{0+1+2+3=6}{3+3+3+3=12} = \frac{1}{2}$, $\frac{0+1+2+3+4=10}{4+4+4+4+4=20} = \frac{1}{2}$. $\frac{0+1+2+3+4+5=15}{5+5+5+5+5+5=30} = \frac{1}{2}$ $\frac{0+1+2+3+4+5+6=21}{6+6+6+6+6+6+6=42} = \frac{1}{2}$.
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Figure 2. John Wallis' sums

John Wallis eventually arrived at the following rule "*per modum inductionis*." This was centuries before the logic for a proof by induction was in place. The rule has a geometric context to it, and can seem bizzarre taken out of context. The problem at hand was essentially what is $\int x^2$ and in general what is $\int x^n$ using a discrete approach (pun intended!).

$$\frac{0+1}{1+1} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{0+1+4}{4+4+4} = \frac{1}{3} + \frac{1}{12}$$

$$\frac{0+1+4+9}{9+9+9+9} = \frac{1}{3} + \frac{1}{18}$$

$$\frac{0+1+4+9+16}{16+16+16+16} = \frac{1}{3} + \frac{1}{24}$$

$$\frac{0+1+4+9+16+25}{25+25+25+25+25} = \frac{1}{3} + \frac{1}{30}$$

$$\frac{0+1+4+9+16+25+36}{36+36+36+36+36+36} = \frac{1}{3} + \frac{1}{36}$$

$$\frac{0^2+1^2+2^2+\cdots n^2}{n^2+n^2+n^2+\cdots n^2} = \frac{0+1+4+\cdots n^2}{n^2+n^2+n^2+\cdots n^2} = \frac{1}{3} + \frac{1}{6n}$$

1. The creation or new tools/techniques that are hitherto unknown to the field or the adaptation of what is known, and their confrontation with the gatekeepers of the field (labelled **Risk taking**).
2. The convergence or divergence of the new theorem/technique/result with other results in that particular area of research or inquiry (that lend **Coherence**) and can lead to other areas of research.

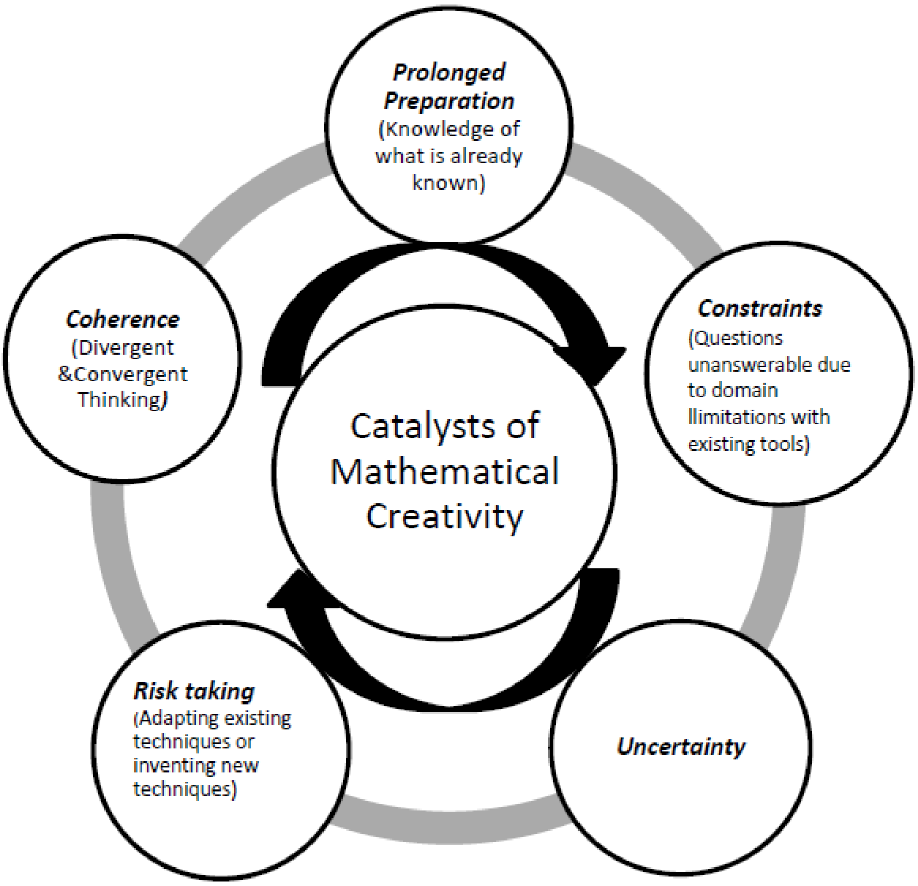


Figure 3. The Catalysts of Mathematical Creativity

These five factors are captured in terms of a cyclical process that catalyses mathematical creativity (in Figure 3) under the designated labels. In other words given a mathematical problem or exploration, prolonged preparation makes the mathematician aware of the constraints related to the problem both in terms of the available tools/techniques and as well as any constraints imposed by known results. This leads to the grey area of uncertainty, which is overcome by risk taking which in some cases leads to a solution to the problem or the creation of a new technique or a new result that ultimately needs to cohere with existing results in the area of investigation and in some cases lead to new areas of investigation.

The case of Louis de Branges' proof of the Bierberbach conjecture (posed by Ludwig Bieberbach in 1916) can serve as an excellent illustration of this cyclical process and an extreme example of practices that occur among professional mathematicians with respect to long standing unsolved problems that bestow prestige to the solver. It also is a tale of self-efficacy in the face of adversity. Succinctly put, in 1984 when de Branges announced he had a proof for the (then) nearly 70-year old conjecture, he was ignored by the U.S. community of mathematicians. Most of the U.S. mathematicians he sent his manuscript to, were not bothered to read it. The ones who began reading it found some minor errors initially which stopped them from reading further based on his "professional reputation." In fact de Branges had been shunned by the U.S mathematics community for nearly 30 years and labelled as someone who gave incorrect proofs. This notoriety had occurred because he had published an incorrect proof early in his career, and exacerbated by his subsequent announcements of having solved open problems, which were also deemed incorrect. This particular conjecture had been worked on by prominent mathematicians like Charles Loewner, J.E. Littlewood, I.M. Milin (who had posed a stronger version of the conjecture) and others. The 350 page manuscript sent by de Branges with his proof of the Bieberbach conjecture (in fact of Milin's stronger version of the conjecture) also consisted of unorthodox methods not normally used in analysis. However a fortuitous exchange agreement between the U.S and the Soviet Union in 1984, allowed him to travel to the Soviet Academy of Sciences where he found a sympathetic but sceptical audience amongst I.M. Milin and his colleagues at the University of Leningrad.

Milin and his colleagues sat through five lectures over five days given by de Branges that lasted a total of 20 hours, at the end of which they confirmed to the community that de Branges was correct! With the help of his Soviet colleagues at the Steklov Mathematics Institute in Leningrad, he published a reduced 12 page preprint that contained the basic ingredients of his methods and the proof (de Branges, 1984) and the following year a paper with his proof appeared in the prestigious international journal *Acta Math*. This case has all the ingredients of the 5 factors that catalyse mathematical creativity- a mathematician with the necessary knowledge but a damaged reputation, a problem that had known constraints in the area of Complex Analysis which had been worked on by many generations of prominent mathematicians who had created different methods. Uncertainty played a pivotal role since many believed the conjecture was probably false. When solving this problem, de Branges had verified his conjectured inequality numerically for the first 30 coefficients with colleagues at Purdue University. His proof's ultimate validation by I.M. Milin led to recognition and acknowledgement from mathematicians that had shunned him, and also allowed for coherence to occur with previous approaches that involved particular cases and different methods/techniques invented that today are their own areas of mathematical research. In the remainder of the paper, several episodes from mathematics in relation to the development of the Logarithm as a computational tool by Napier, and the equivalence of representations of numbers constructed by Euler and Lambert are used to demonstrate this cyclical process. The examples have been carefully chosen to highlight the creative and ingenious use of arithmetic for solving the particular problems at hand.

EPISODES FROM MATHEMATICS

As stated earlier, the three episodes from mathematics that are now chosen to illustrate the theoretical framework and in particular demonstrate ingenuity in the use of arithmetic since no other tools/methods were available in these time periods.

Episode 1

Logarithms are an essence of both mathematical invention and an example of ingenuity to overcome computational constrains posed by problems in post-medieval astronomy. The numbers arising from trigonometric functions involved 7 to 8 digits. Astronomers used clever trigonometric identities to convert multiplication problems into those involving addition. Trigonometric identities like $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$ were useful for multiplying large numbers with the use of sine tables.

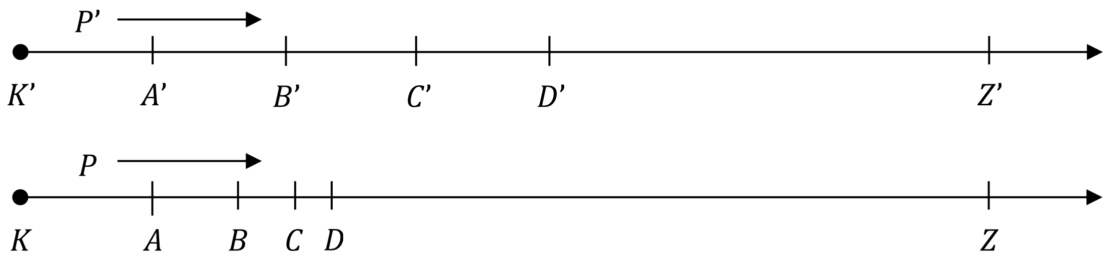
Logarithms today are simply conceived of as function $\mathcal{L}(x)$ such that $\mathcal{L}(xy) = \mathcal{L}(x) + \mathcal{L}(y)$ for all $x, y > 0$. Delving into their conception as done by John Napier offers us a true example of the act of creating something new out of something existing. Recall arithmetic and geometric progressions were known since the time of Euclid. Napier's ingenuity to create artificial numbers out of them can be understood as follows.

In the Constructio (1619) published posthumously this is described as follows:

Definition 1: A line is said to increase equally when the point describing the same goeth forward equal spaces in equal times.

Definition 2: A line is said to decrease proportionally into a shorter when the point describing the same in equal times cutteth off parts continually of the same proportion to the lines from which they are cut off.

Gibson (1915) in the *Napier Tercentenary Memorial Volume* provides a clear insight into Napier's conception by providing Napier's original work:



$A'Z'$ is a line at unlimited length increasing equally by the motion of P'.

AZ is a line of finite but large length ($= 10^7 units$) which is decreasing proportionally by the motion of P.

So, we have:

$$AB{:}AZ = BC{:}BZ = CD{:}CZ \text{ ... and } A'B' = B'C' = C'D' = \cdots$$

At any instant of time the number that "measures" $A'P'$ is the logarithm of the number that measures PZ or

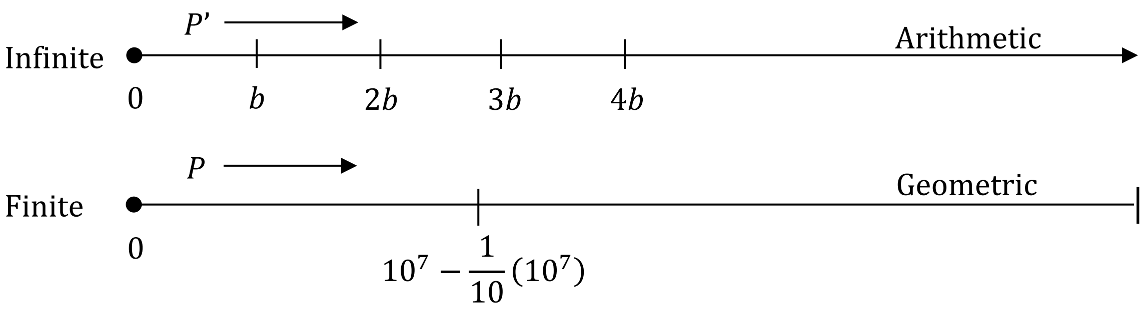
$$A'P' = \log PZ$$

$$A'B' = \log BZ$$

$$A'C' = \log CZ \text{ etc.}$$

Napier has been written about extensively. However what is not elucidated enough is that this contains both the seeds of differential equations as well as "interval arithmetic" (which only developed formally in the 1950s). The differential approach to Napier's logarithm shown in books today actually trivializes Napier's conception, hence we will investigate his "interval arithmetic" approach which are truer to his propositions and less well known.

First, the intervals are worth exploring to gain an insight into Napier's invention:



For the sake of experimentation let the geometric progression have a ratio of $\frac{1}{10}$.

Then, we can start to mark out points α, β, γ, etc. on the finite line as follows:

$$\alpha = 10^7 - \frac{1}{10}(10^7) = 10^7 - 10^6$$

$$\beta = 10^7 - \frac{1}{100}(10^7) = 10^7 - 10^5$$

$$\gamma = 10^7 - \frac{1}{1{,}000}(10^7) = 10^7 - 10^4$$

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$$\tau = 10^7 - \frac{1}{1{,}000{,}000}(10^7) = 10^7 - 10^6$$

$$\varphi = 10^7 - \frac{1}{10{,}000{,}000}(10^7) = 10^7 - 1$$

In seven iterations we have only come one unit within α but what happens after that?

$$\theta = 10^7 - \frac{1}{10^8}(10^7)$$

$$\iota = 10^7 - \frac{1}{10^9}(10^7)$$

$$\kappa = 10^7 - \frac{1}{10^{10}}(10^7)$$

$$\mu = 10^7 - \frac{1}{10^{11}}(10^7)$$

These terms $\theta, \iota, \kappa, \mu$ get very, very close to 10^7. What is the upshot of this calculation? The intervals in the geometric progression become smaller and smaller ad infinitum <u>but</u> they can be stretched out to have equal length on the infinite line and point P' can travel between these stretched points in an increment of time, say 1 second, then 2 seconds, 3 seconds etc.

Are we are able to determine exponents of a given number to a chosen base i.e. given a number α and a base say 10, can we determine a number "N" such that $\alpha = 10^N$? In modern terms we would have the natural numbers 1, 2, 3, on the infinite line, and the powers of 10 on the finite line. However this is not quite the case in Napier's formulation. The arithmetic-geometric conception of Napier has subtleties that only become evident when we set the ratios as defined by him in the six propositions of the Descriptio (1614). One cannot simply read out the exponents to get the logarithm as a power of 10 or e (as we do today) but have to calculate it using interval arithmetic. The following propositions are from Gibson (1915).

Proposition 1 ≔

The logarithms of proportional numbers and quantities are equally differing, i.e.

if $BZ : CZ = DZ : EZ$,

then $\log CZ - \log BZ = \log EZ - \log DZ$

Note: $\log CZ = A'C'$ *and* $\log BZ = A'B'$ etc.

Proposition 2 ≔

if $a : b = b : c$,

then $\log c = 2 \log b - \log a$

which is a consequence of Prop. 1 since

$\log b - \log a = \log c - \log b$.

Proposition 3 ≔

if $a : b = b : c$,

then $2 \log b = \log a + \log c$

Proposition 4 ≔

if $a : b = c : d$,

then $\log d = \log b + \log c - \log a$.

This leads one to think that the modern rules of logarithms are being followed and that

$$\log \frac{bc}{a} = \log b + \log c - \log a$$

But

$$\log bc \neq \log b + \log c$$

and

$$\log \frac{b}{a} \neq \log b - \log a.$$

To use Napier's system means to write

$x = bc$

and form a proportion

$\frac{x}{b} = \frac{c}{1}$

then

$\log x = \log b + \log c - \log 1.$

But $\log 1 \neq 0$ in Napier's original conception, which will be explored further shortly.

Now, if we denote some number "r" whose log is zero, then starting with $x = \frac{bc}{r}$, we get

$$\log x = \log b + \log c - \log r$$
$$= \log b + \log c$$

Gibson (1915) in his exposition implicitly asks the question. How can we force the issue forward to arrive at the only creative and necessary conclusion that $\log 1$ is best set as zero?

Using Proposition 1 we can write with algebraic notation $\log ak - \log bk = \log a - \log b$ since $\frac{a}{k} = \frac{b}{k}$

then

$\log ac - \log bd = (\log ac - \log bc) + (\log bc - \log bd) \quad = (\log a - \log b) + (\log c - \log d).$

If $a = c$ and $b = d$

we get,

$\log a^2 - \log b^2 = \log a - \log b + \log a - \log b = 2(\log a - \log b)$

Pushing this forward, we can arrive at

$\log a^k - \log b^k = n(\log a - \log b).$

Now if we compute powers of 10 to connect a number "a" and $10^n a$, we get
$\log a - \log 10a = \log 1 - \log 10$

since

$\frac{a}{1} = \frac{10a}{10}.$

Napier set $\log 1 = 7 \log 10^6$ and $\log 10^7 = 0$. This meant

$$\log a - \log 10a = \log 1 - \log 10 = \log 10^6$$

since

$\frac{1}{10} = \frac{10^6}{10^7}$ and $\log 10^7 = 0.$

Now computing other powers of 10 in relation to "a" yields $\log a - \log 100a = 2(\log 1 - \log 10) = 2\log 10^6$

$\log a - \log 1{,}000a = 3(\log 1 - \log 10) = 3\log 10^6$

and in general

$$\log a - \log 10^n\, a = n(\log 1 - \log 10) = n \log 10^6$$

Clearly the relationship between the logarithms of a and $10^n a$ is more complicated than desired. The ratio of 10 to 1 does not yield 1 as the logarithm of 10, i.e., $\log \frac{10}{1} \neq 1$.

Fixing $\log \frac{10}{1} = 1$ would yield $\log \frac{100}{1} = 2$ and $\log \frac{1000}{1} = 3$ and so on…

Briggs, who became familiar with Napier's remarkable system, suggested the adjustment of the scale so that the relationship between the logarithm of a and the logarithms of $10^n a$ were simpler. Setting $\log r = \log 10^{10} = 0$ and $\log \frac{r}{10} = \log 10^9 = 10^{10}$, the relations between the logarithms of a and $10^n a$ become $\log a - \log 10^n a = n(\log 1 - \log 10) = n \cdot 10^{10}$ and in general $\log 10^n = (10 - n)10^{10}$.

This new system made computation simpler, but it was Napier's suggestion that $\log 1 = 0$ that made logarithms (or artificial numbers as Napier called them) increase with the natural numbers as opposed to decrease[7]. While the axioms that defined Napier's logarithms are easy to understand, his ingenuity lay in the hand calculations performed to extract the logarithms using different geometric sequences.

These sequences started at 10^7 and used common ratios of 0.9999999, 0.99999, and 0.9995 respectively. In the first case, he computed 101 terms to seven decimal places by starting with 10^7, using $1 - \frac{1}{10^7} = 0.9999999$ as the common ratio, and successively subtracting this from the preceding number. i.e.

$a_0 = 10{,}000{,}000.0000000$ and $\mathcal{L}(a_0) = 0.0000000$

$a_1 = \;\; 9{,}999{,}999.0000000$ and $\mathcal{L}(a_1) = 1.0000001$

$a_2 = \;\; 9{,}999{,}998.0000001$ and $\mathcal{L}(a_2) = 2.0000001$

$a_3 = \;\; 9{,}999{,}997.0000003$ and $\mathcal{L}(a_3) = 3.0000002$

.

.

.

$a_{100} = 99{,}999.0004950$ and $\mathcal{L}(a_{100}) = 100.0000050$

[7] There is some debate on whether it was Briggs who made the suggestion. However a close reading of In C.G. Knott's (1915) *Napier Tercentary Volume* which includes papers from authors that have examined the original sources, indicates attribution to Napier.

First Table

$$\left\{10^7\left(1-\frac{1}{10^7}\right)^r,\ r=0 \text{ to } 100\right\}$$

```
10000000·0000000
       1·0000000
----------------
 9999999·0000000
        ·9999999
----------------
 9999998·0000001
        ·9999998
----------------
 9999997·0000003
        ·9999997
----------------
 9999996·0000006
```

to be continued up to

9999900·0004950

Figure 4. Napier's First Table (Hobson, 1914, p.29)

It is important to note that Napier determined the logarithms of these numbers by hand computing bounds for each value in the sequence using what is termed today as "interval arithmetic." To compute the logarithm of 9999999, he determined the lower and upper bounds of $\mathcal{L}(9999999)$ as 1 and $\frac{10^7}{9999999}$ and took their mean 1.00000005 as the logarithm. Hobson (1914) describes his calculation as follows:

> Having formed these tables, Napier proceeds to obtain with sufficient approximation the logarithms of the numbers in the tables. For this purpose his theorems (1) and (2) as to the limits of logarithms are sufficient. In the first table, the logarithm of 9999999 is, in accordance with (1), between 1·0000001 and 1·0000000, and Napier takes the arithmetic mean 1·00000005 for the required logarithm. The logarithm of the next sine in the table is between 2·0000002 and 2·0000000; for this he takes 2·00000010, for the next sine 3·00000015, and so on.

Episode 2

We fast forward a century and a half to the 18th century to study another example of domain constraints spurring innovation from Euler in the existing mathematical techniques. John Wallis' representation of π derived from manipulating tables is found in *Artihmetica Infinitorum* (1655). Wallis' representation was

$$\frac{4}{\pi} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot\cdot 8 \cdot 8 \cdot 10}$$

This was manipulated by Lord Brouncker to derive

$$\frac{4}{\pi} = 1 + \cfrac{1 \cdot 1}{2 + \cfrac{3 \cdot 3}{2 + \cfrac{5 \cdot 5}{2 + etc.}}}$$

For Euler, manipulation of identities was second nature using very fundamental techniques involving arithmetic. Euler's manipulation of infinite series was groundbreaking work that did not conform to the rules of analysis that would be formalized much later. Another representation of $\frac{4}{\pi}$ in the form of an alternating series was available at this time. It is also known as the Leibniz series (and is the reciprocal of $\frac{4}{\pi}$) namely

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \mp \cdots$$

Using very elementary arithmetic, Euler transformed alternating series to continued functions as follows:

Suppose

$$l = \frac{1}{a} - \frac{1}{b} + \frac{1}{c} - \frac{1}{d} + \frac{1}{e} - \frac{1}{f} + etc.$$

Let

$$m = \frac{1}{b} - \frac{1}{c} + \frac{1}{d} - \frac{1}{e} + \frac{1}{f} - etc.$$

Let

$$n = \frac{1}{c} - \frac{1}{d} + \frac{1}{e} - \frac{1}{f} + etc.$$

Then,

$$l = \frac{1}{a} - m = \frac{1 - am}{a}$$

$$\therefore \quad \frac{1}{l} = \frac{a}{1 - am} = \frac{a - a^2m + a^2m}{1 - am} = \frac{a(1 - am)}{1 - am} + \frac{a^2m}{1 - am}$$

$$= a + \frac{a^2m}{1 - am}$$

$$= a + \frac{a^2}{-a + \frac{1}{m}}.$$

Similarly,

$$\frac{1}{m}$$

could be rewritten as

$$\frac{1}{m} = b + \frac{b^2}{-b + \frac{1}{n}}$$

and

$$\frac{1}{n} = c + \frac{c^2}{-c + etc.}$$

By rewriting these numbers, we get

$$\frac{1}{l} = a + \cfrac{a^2}{b - a + \cfrac{b^2}{c - b + \cfrac{c^2}{d - c + \cfrac{d^2}{etc.}}}}$$

If we apply this to

$$l = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \mp etc. = \frac{\pi}{4}$$

then

$$a = 1, b = 3, c = 5, d = 7, etc.$$

$$\therefore \quad \frac{1}{l} = \frac{4}{\pi} = 1 + \cfrac{1 \cdot 1}{2 + \cfrac{3 \cdot 3}{2 + \cfrac{5 \cdot 5}{2 + \cfrac{7 \cdot 7}{2 + etc.}}}}$$

...and is precisely the representation of Lord Brouncker!

Thus, one sees elementary arithmetic and interpolation being used to switch between the representation of $\frac{4}{\pi}$.

Episode 3

The third example of an astonishing manipulation of elementary arithmetic is the continuous function of $\tan x$ derived by Lambert, a Swiss contemporary of Euler in the 1760s.

$$\tan x = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}}$$

$$= \frac{x\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right)}{(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}}$$

$$= \frac{x}{\frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right)}{\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right)}}$$

$$= \frac{x}{1 - \frac{x^2\left(\frac{1}{3} - \frac{x^2}{30} + \frac{x^6}{5040}\right)}{\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right)}}$$

$$= \frac{x}{1 - \frac{x^2}{\left(\frac{1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}}{\frac{1}{3} - \frac{x^2}{30} + \frac{x^6}{840}}\right)}}$$

As $x \to 0$, the denominator $\to 3$, factor out $\frac{1}{3}$

$$= \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{etc.}}}$$

This suggested to Lambert that the continued function for $\tan x$ must be

$$\frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5\ etc.}}}$$

Note that he did not compute more than 3 steps of his “divide and factor out” arithmetic.

It was Legendre (1794) who actually proved that this representation was in fact correct by considering

$$f(z) = 1 + \frac{a}{1 \cdot z} + \frac{a^2}{1 \cdot 2 \cdot z(z+1)} + \frac{a^3}{1 \cdot 2 \cdot 3 \cdot z(z+1)(z+2)} + \cdots$$

And defining

$$g(z) = a \cdot \frac{f(z+1)}{z \cdot f(z)}$$

which on iteration leads to Lambert's representation provided $a = \frac{x^2}{4}$.

Once the theory of convergents for continued fractions as developed by Euler in 1748 is applied to Lambert's representation of $\tan x$, the partial quotients for the continued fraction expansion of make the end result more palpable and cohere with the end result.

The first four partial quotients of the continued fraction are

$$\frac{x}{1};\ \frac{3x}{3-x^2};\ \frac{15x-x^3}{15-6x^2};\ \frac{105x-10x^3}{105-45x^2+x^4}$$

Again, Lambert inspired by Euler, used elementary arithmetic to arrive at his representation.

DISCUSSION

In this section, the first two episodes are discussed through the lens of the theoretical framework for creativity (namely the 5 principles) as well as the catalysts of mathematical creativity.

Discussion of Episode 1

John Napier lived in an era where the construction of tables for trigonometric functions was one of the major problems of that time period with profound implications for astronomy. Hobson (1914) called the second half of the 16th century "the age of numerical calculation." Seen through the lens of our theoretical framework, Napier's prolonged preparation for the task that occupied a major portion of his life consisted of knowledge of arithmetic and algebra of that time period including general methods for the extraction of roots of numbers of higher degrees. The development of the logarithm relied on notions of arithmetic and geometric progressions as we have seen in his definitions and propositions. However the "Big C" creativity lay in what Hobson (1914) appropriately called the invention of representations of numbers as continuously moving points and "the conception of a functional relationship between two continuous variables" (p.45). Moreover the numerical approximations of the functional relationship contained the seeds of "interval arithmetic." Formal methods of interval arithmetic that bridged mathematics with the developing field of computing only developed 350 years later! Napier's invention of *numerus artificialis* (logarithm) was supported by Henry Briggs, a Professor of Geometry at Gresham College in London, who made the arduous journey to visit Napier in Scotland. Over time the cordial relationship between these two gentlemen led to the improvement of Napier's logarithm (change of its base to 10) which made computation much easier.

In spite of the divergence of Napier's new conception of *numerus artificialis* (artificial numbers) with existing notions of number, the use of arithmetic and geometric progressions to arrive at this invention allowed Brigg's to understand the methods devised by Napier for the construction of his tables. Coherence was achieved as a result of Brigg's collaboration, his validation and subsequent improvement of the invention of Napier. The catalysts for mathematical creativity during Napier's time period were the necessity to have accurate trigonometric tables to facilitate numerical calculation for astronomers, and the computational constraints imposed by the known methods at that time period. This served as an impetus for Napier to embark on a 20 year computational odyssey that changed the nature of mathematical computation for astronomers and mathematicians in the centuries that followed. Napier was in a position as the *Laird of Merchiston* to have the freedom of time (the Gestalt principle in Sriraman's theoretical model), the disposition to tackle a problem that was vexing for that time period (the Scholarly principle) and to persevere in spite of the uncertainty in the methods that he had invented to construct artificial numbers. The mathematical constraints of the time period called for ingenuity in methods which spurred creativity. Haught and Stoke (2017) argue that domains are defined by constraints and creativity can be viewed as mastering the constraints and developing the necessary tools/techniques to achieve the desired goals in spite of the existing limitations.

Discussion of Episode 2

In the same vein, domain limitations in the "quadrature problem," viz., computing areas under curves long before having the Binomial theorem with rational exponents and the subsequent invention of Calculus led John Wallis to devise a table that gave values of $\frac{1}{\int_0^1 \left(1-x^{\frac{1}{Q}}\right)^P dx}$ for convenient values of P and Q. The goal was to compute the quadrature of a unit circle, namely $\int_0^1 \sqrt{1-x^2}\,dx$ by interpolating values obtained from generating tables for $\int_0^1 (1-x^2)^0 dx$ and $\int_0^1 (1-x^2)\,dx$. Reciprocal values were obtained for ease of computation. The numerical value obtained by Wallis was $\frac{4}{\pi} = \left(\frac{3}{2}\right)\left(\frac{3\cdot5}{4\cdot4}\right)\left(\frac{5\cdot7}{6\cdot6}\right) \ldots$ (the answer to the quadrature was the reciprocal, namely $\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\,dx$). The sharing of these tables by John Wallis with Lord Brouncker led the latter to an entirely different interpretation to solve the quadrature problem, and a completely different representation, namely:

$$\frac{4}{\pi} = 1 + \cfrac{1\cdot1}{2 + \cfrac{3\cdot3}{2 + \cfrac{5\cdot5}{2 + etc.}}}$$

The other representation of this number (strictly speaking its reciprocal) from that time period was

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - + \cdots$$

Episode 2 described the technique devised by Euler to convert an alternating series to a continued fraction that allows one to see that the expression obtained by Lord Brouncker could be obtained from the Leibniz series. Euler developed more general methods to switch between continued fractions and infinite series and vice versa, in addition to a theory of convergents. This astonishing manipulation with the use of basic arithmetic is another example of "Big C" creativity.

Euler needs no introduction in the mathematics community- his original work is vast, ground breaking and ubiquitous with genius needing no further expostulation. However the purpose of including a small example from his work is to address the catalysts of mathematical creativity in the face of uncertainty. During Euler's time period arithmetic had advanced to now include the arithmetic of series which were treated informally as infinite polynomials. Domain constraints from the field on the arithmetic of infinite series (in the form of "rules") only came into place when the theory of Analysis was made formal/rigorous nearly a century after the time of Euler. In terms of what was possible to achieve by using the arithmetic of series (including series inversion), Euler's work provides numerous examples of "Big C" creativity, namely the use of existing techniques and their adaptation to solve problems that lent coherence to many areas of mathematics unhindered by gatekeepers. In this case, the uncertainty of the equivalence of different representations of the same number is removed lending coherence. For Euler, "risk taking" was second nature particularly in the arithmetic of series- and since he lived in a time period where the rules of what was allowed were unclear, it allowed for existing mathematical techniques to be used in creative ways that would not be possible today. Finally, Episode 3 can also be analyzed using the provided framework.

CONCLUDING POINTS

In this paper three examples from mathematics were given to illustrate how uncertainty in the form of constraints can take on different forms. In the case of Napier, computational constraints imposed by the use of trigonometric tables served as a catalyst for the development of the logarithm and an entirely new arithmetic developed from what was known about arithmetic and geometric progressions. Napier's insight into art of interpolation and interval arithmetic created new tools that were of tremendous value to the world. In the case of Euler, the existing impasse reached on particular problems, were overcome with the insight of adapting existing tools to go beyond the (perceived) constraints of the problem. In this process one notes that for Euler arithmetic was a tool that could be put to use creatively in a variety of problems. In these cases from mathematics, both Napier and Euler did not have to experience the frustration from the field not accepting their methods used to overcome the constraints. Recall the 20th century case of de Branges elucidated earlier in this paper. This was a result of their timing in history where "rules" from gatekeepers had not shackled mathematicians, as well as the blessing of not having too many tools/techniques to work with, namely the limitations of the existing mathematics of their time period. An old adage is applicable here: it states that when a child learns to use a hammer, everything begins to look like a nail.

The versatile and creative use of arithmetic to create new mathematics, as well as to lend coherence to the existing mathematics provides us with a lesson. Namely, constraints in the form of limited tool kits (and not in the form of rules imposed by the field) can overcome uncertainty and spur creativity. The question is whether this is teachable in the classroom to foster the catalysts of creativity that are present at the professional level? There are several avenues to achieve this. Zaslavsky (2005) suggests a focus on mathematical task design that includes the component of uncertainty, and redesigning the task based on the outcomes of the implementation in the classroom. Brown and Walter (2005) in their classic book *The Art of Problem Posing* provide the heuristic of "what if-not" to analyze attributes in a problem to either tighten or loosen constraints and in many cases pose problems of a more general or ambiguous nature that may or may not be solvable. In recent years, several studies have looked at methods for stimulating mathematical creativity in the classroom. Beghetto and Schreiber (2017) proposed abductive reasoning as another approach to stimulating creativity, which represents a form of creative reasoning triggered by states of genuine doubt. Via abductive reasoning, we resolve our doubt and this in turn can be a key motivator in the creative learning process. Examples of creating doubt, closely related to abductive reasoning are found in Sriraman and Dickman (2017) in which the use of mathematical pathologies as a means of fostering creativity in the classroom is proposed. Last but not least, the history of mathematics provides many tasks that can be posed with the caveat that only time period methods are permissible to students, i.e., constraints imposed with restricted tool-kits. Students of mathematics will have to find creative ways to use their knowledge to solve such problems. The episodes in this paper show that such constraints led to astonishing adaptation and creative use of arithmetic that spurred the development of the field of mathematics.

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THEORETICAL CONSIDERATIONS

OVEREXCITABILITY, ICONOCLASM AND MATHEMATICAL CREATIVITY & GIFTEDNESS

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Abstract. *There are several theoretical psychological concepts in the realm of research on (mathematical) creativity and giftedness, e.g. originality, non-conformism, iconoclasm, overexcitability and high sensitivity. By connecting these aspects to one another we show some concept-immanent interdependencies and congruities. Applying those to the specific area of mathematics we identify a natural relation of the mentioned concepts to the character of performing and dealing with mathematics. Additionally, we derive some consequences for classroom teaching.*

Key words: overexcitability, iconoclasm, originality, high sensitivity, creativity, giftedness, mathematics.

THE CONNECTION OF SEVERAL PSYCHOLOGICAL ASPECTS

We present a theoretical attempt of bringing together several psychological aspects that are related to creativity and giftedness. Concentrating on the very basis of the neural origins of mathematical giftedness and creativity, and investigating how that might be interpreted in the context of psychological characteristics of a highly sensitive personality, leads to consequences for classroom teaching and concepts of fostering.

Creativity, originality, non-conformism and iconoclasm in a mathematical context

According to Torrance (1966) there are four related components of *creativity*: fluency, flexibility, originality and elaboration. Among others, Leikin and Kloss (2011) emphasized the special mental quality of originality in the context of mathematical problem solving. Within the effort of testing these dimensions, by using the Torrance Test of Creative Thinking, originality is delimited by the statistical rarity of the responses. Or, as James (1999-2000) puts it: "Something is created that has never before existed in exactly that form." (p. 115). The composing factors of the Composite Creative Personality (CCP) scale (Harrington 1972, 1975) – as cited in Barron & Harrington (1981, p. 454) – implies a personality which, among other characteristics, is described as assertive, ambitious, idealistic, independent, individualistic, original, rebellious, self-confident, sensitive, spontaneous, versatile, *not* conventional and *not* inhibited. That is, a truly non-conformist personality. Similar descriptions can be found in Runco (2014), portraying a creative person as "independent, non-conformist, rebellious, unconventional, norm-doubting, and contrarian" (cited in Mann et al., 2017, p. 64). Importantly, Mann et al. (2017), drawing on Hadamard's work, suggest that intuitive problem solvers, among other essential mathematical abilities, also need the courage to reveal and share their results with their peers or within the mathematical community. Thus, focusing on the juncture between affect and creativity, Mann et al. propose an extension of the widely used creativity model (fluency, flexibility, originality and elaboration) with the concept of *iconoclasm* – defined as an affective and non-conformist state of mind necessary for the revelation of an original, creative product.

Overexcitability (OE) and the Highly Sensitive Person (HSP)

One of the items on Harrington's (1972, 1975) CCP scale is a certain kind of *sensitivity* within the creative person. Emphasizing sensitivity translates to dealing with a significantly distinct sensitivity of an individual's neural system. There are different approaches and terms when addressing this context. Mendaglio & Tillier (2006) sum up the research literature belonging to the Theory of Positive Disintegration, described by Kazimierz Dabrowski between 1964 and 1972, by noting that his non-mainstream theory of personality development assumes that psychological tension and anxiety are necessary for developing true individuality. Dabrowski (1972) defined (psychic) *overexcitability* (OE) as "higher than average responsiveness to stimuli, manifested either by psychomotor, sensual, emotional (affective), imaginational, or intellectual excitability, or the combination thereof" (p. 303, cited in Mendaglio & Tillier, 2006, p. 69). Mendaglio and Tillier refer to Piechowski's work during 1975-1991, as for the elaboration of the conceptual basis of overexcitability, which was put more precisely by Tucker & Hafenstein (1997) by connecting it to Dabrowski's original descriptions. Besides the two basic physiological components, **Psychomotor OE** and **Sensual OE**, they describe the remaining three manifestations of OE as follows:

"**Intellectual Overexcitability**. The manifestations of intellectual overexcitability are associated with an intensified and accelerated activity of the mind. Its strongest expressions have more to do with striving for understanding, probing the unknown, and love of truth than with learning per se …

Imaginational Overexcitability. The presence of imaginational overexcitability can be inferred from frequent distraction, wandering attention, and daydreaming. These occur as consequence of free play of the imagination. Here, too, belong illusions, animistic thinking, expressive image and metaphor, invention …

Emotional Overexcitability. Among the five forms of psychic overexcitability, the manifestations of emotional overexcitability are the most numerous. They include certain characteristic and easily recognizable somatic expressions, extremes of feeling, inhibition, strong affective memory, concern with death, anxieties, fears, feeling of guilt, and depressive and suicidal moods. …" (p. 68, boldface and ellipses in original, as cited in Mendaglio & Tillier, 2006, pp. 70–71)

Another concept dealing with an intensified physiological experience of sensory stimuli is the work done on *highly sensitive persons* (HSPs), described in Aron (1996), Aron & Aron (1997) and Trappmann-Korr (2010). Aron (1996) phrases this trait as "having a sensitive nervous system […] probably inherited" (p. xiii) and continues: "It occurs in about 15-20 percent of the population. It means you are aware of subtleties in your surroundings, a great advantage in many situations. It also means you are more easily overwhelmed when you have been out in a highly stimulating environment for too long, bombarded by sights and sounds until you are exhausted in a nervous-system sort of way." Based on an interview study, a follow-up questionnaire and a randomized telephone survey (Aron & Aron, 1997), it is shown that "introversion and sensitivity were being wrongly equated" (p. xix) and the presented framework is called sensory processing sensitivity (SPS).

Trappmann-Korr (2010) also discusses the five components of OE with respect to Dabrowski – and especially to Webb et al. (2005) – but delivers a different declaration (probably depending on the German translation) of imaginational OE, as a "**creative overexcitability (imagination/fantasy)"**. She points out that these individuals love symbols in early ages and are able to enter an imaginational world, where they play with virtual things. These remarks can also be related to Krutetskii's (1976) description of the mathematical cast of mind of mathematically gifted pupils (pp. 302–305) and to their distinctive aesthetic, joyful feelings associated with mathematical problem solving (p. 347).

The aim of this paper

Based on the above presented theories of originality, creativity, non-conformism and iconoclasm, and about the neural basis of the psychological characteristics of OE and the HSP, we would like to investigate possible implications for classroom teaching in order to develop mathematical giftedness and creativity.

Findings

Connections between giftedness, creativity, overexcitability and the HSP

Rinn et al. (2010) point out that "multiple researchers have found that gifted individuals tend to score higher than the nongifted on some forms of the overexcitabilities. Piechowski and Colangelo (1984) examined the overexcitabilities of intellectually gifted adolescents, intellectually gifted adults, artists, and average-ability graduate students. These results indicate that both gifted adolescents and gifted adults were characterized by higher intellectual, emotional, and imaginational overexcitabilities. Gallagher (1986) found that gifted 6th-grade students also reported higher intellectual, emotional, and imaginational overexcitability scores than a random sample of average-ability 6-grade students. In a study of 10th- and 11th-grade students, gifted students had higher intellectual, emotional, and psychomotor overexcitabilities than average ability students (Ackerman, 1997), and Bouchet and Falk (2001) found gifted college students to score higher than average-ability college students on measures of intellectual and emotional overexcitabilities." (p. 4–5)

In a mathematical perspective of creativity, Mann et al. (2017) state that iconoclastic problem solvers are not satisfied with "good enough" results and seek for more aesthetic and efficient solutions. Noting that iconoclasm might be suppressed by anxiety or authoritarian environments, the authors advocate for an educational environment which is encouraging to share their findings in order to develop their creativity.

In addition, in a speech about the history of creativity research at the conference "Empowering Creativity in Education" in Utrecht, Mark A. Runco (2016), emphasized that there is an immanent contradiction between the essential part of originality within the construct of creativity and a traditional classroom setting established by a strict curriculum guidance and non-open teaching formats with single solution tasks for all learners. He stated that such an outdated way of teaching supports a conformist behaviour on the part of the learners. Conversely, thinking in an original way implies to be non-conformist, as being a conformist means to orientate on the statistical majority of a peer group, which is the direct opposite of representing a high level of originality traits.

In a discourse addressed to teachers, about students who are HSP, Aron (1996) states that:

- "HSPs are generally conscientious and try their best. Many of them are gifted. But no one performs well when overaroused, and HSPs are overaroused more easily than others. The harder they try when being observed or otherwise under pressure, the more they are likely to fail, which can be very demoralizing for them.
- High levels of stimulation (e.g., a noisy classroom) will distress and exhaust HSPs sooner than others. While some will withdraw, a significant number of boys especially will become hyperactive.
- Don't overprotect the sensitive student, but when insisting that the student try what is difficult, see that the experience is successful.
- Do make allowances for the trait while the student is gaining social stamina. If a presentation is to be made, arrange for a "dress rehearsal" or the use of notes or reading aloud – whatever lowers the arousal and permits a successful experience.
- Do not assume that a student who is just watching is shy or afraid. It may be quite the wrong explanation, yet the label may stick …
- Watch for and encourage the creativity and intuition that is typical of HSPs. To build their tolerance for group life and social status with their peers, try drama activities or dramatic readings of works that have moved them. Or read their work out loud to the class. But be careful not to embarrass them." (pp. 234–235)

Connections between iconoclasm, mathematical giftedness and overexcitability

Mendaglio & Tillier (2006) indicate that Dabrowski's imaginational OE occurs "as consequence of free play of the imagination" (p. 71). According to Lockhart (2009) it holds that: „If there is anything like a unifying aesthetic principle in mathematics, it is this: *simple* and *beautiful*. Mathematicians enjoy thinking about the simplest possible things, and the simplest possible things are *imaginary*. […] I'm just *playing*. That's what math is – wondering, playing, amusing yourself with your imagination." (p. 24). Drawing on the aesthetical aspects of mathematics, Pluess et al. (2017), statistically identified Aesthetic Sensitivity (e.g. being deeply moved by arts and music, having a rich and complex inner life, being conscientious, being aware of subtleties) as one of three factors for the global concept of environmental sensitivity (the other two factors being Low Sensory Threshold and Ease of Excitation). As mentioned, Mann et al. (2017) point out, that iconoclastic problem solvers need the essential courage to reveal their results and a learning environment that permits them to think and work in unconventional ways, moreover, they are not satisfied with "good enough" results and seek for more aesthetic and efficient solutions. The latter characteristic also corresponds with Lockhart's (2009) description of mathematics and with Krutetskii's (1976) description of mathematically gifted pupils. Furthermore, Trappmann-Korr (2010) interprets imaginational OE as a more creative overexcitability by describing an imaginational world, where individuals play with virtual things. Again, his resembles with Lockhart's description of performing mathematics, which, in some way, is a scientific art form of manipulating a language of symbols in an aesthetic way. However, Intellectual OE appears also to be related to the mathematical giftedness construct (e.g. Krutetskii, 1976), as "striving for understanding, probing the unknown, and love of truth" (Mendaglio & Tillier, 2006, pp. 70–71) may be interpreted as a strong urge for deduction and proof – basic concepts of mathematics and mathematical thinking. Also, further aspects of mathematical giftedness are revealed for example in Singer et al. (2013).

Conclusions – consequences for fostering mathematical giftedness and creativity

Based on the presented similarities between the displays of mathematical giftedness and creativity, and, on common characteristics of Intellectual, Imaginational and Emotional OE and HSPs, our aim was to find plausible connections between the concepts, which could have implications for classroom teaching. Based on the presented findings, we would like to suggest that many, but not all, mathematically gifted pupils are at least intellectually OE and/or HSPs. Consequently, it is not unreasonable to assume that these pupils would benefit from a safe, permitting and structured learning environment. In addition, based on studies on mathematical creativity and giftedness, we would like to accentuate the importance of iconoclasm and non-conformism in the mathematics classroom.

In similar ways, Brandl (2014) suggests that mathematically gifted students might "feel some kind of alienated in their old / "normal" classes" and "separating them from their old / "normal" classmates seems to be a promising way to give them an appropriate surrounding for performing and learning mathematics", so – by taking into account additional pedagogical and psychological issues, as those discussed for iconoclastic OE-SPS-HSPs so far – "additional non-performance orientated courses … seem to be suitable settings" (pp. 1163–1164). These suggestions are in concordance with some findings of Szabo's (2017) systematic review of mathematics education for gifted pupils, which indicates that ability grouping outside the heterogenous mathematics classroom is beneficial for and highly appreciated by gifted pupils if the participation in these groups is voluntary, if the teaching is adapted to the pupils' mathematical level and if teachers who work with these groups are well prepared for the characteristics of gifted pupils.

The limitations of this paper do not permit us a deeper, or a broader, discussion of the presented findings. Importantly, our findings are based on a theoretical approach and should be tested through empirical studies. Despite that, we would like to recommend that the teacher and the teaching situation provides a calm, iconoclastic and stable structure, in order to guarantee a safe environment, which allows the sensitive neural system of the gifted pupil to concentrate on essential creative and non-conformist flow-processes.

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THE INTERSECTION OF PROBLEM POSING AND CREATIVITY: A REVIEW

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Abstract. *In this article, we take an in-depth look at research on the intersection of problem posing and creativity in order to present its current state of research in a systematic review. A full search in top journals from mathematics education and the Web of Science revealed only 15 articles from different genres, of which 11 were included in the analysis. Those articles were sorted into two clusters, depending on whether the articles focus on the identification or the fostering of creativity.*

Key words: Problem Posing, Creativity, Review

INTRODUCTION

In the 1990's, Edward Silver published two seminal articles in which he addressed both mathematical problem posing and mathematical creativity. The first article (Silver, 1994) deals with problem posing, emphasizing it as a characteristic of creative activities and mathematical ability. In the second article, Silver (1997) takes the opposite perspective, mainly addressing creativity and highlighting its connections to problem posing (as well as problem solving). Both contributions are widely cited in research literature and constitute the theoretical foundation for many studies dealing in one way or another with problem posing and creativity (cf. Bonotto, 2013; Voica & Singer, 2013; Van Harpen & Presmeg, 2013; Sriraman & Dickman, 2017; Singer, Sheffield, & Leikin, 2017). In a recent handbook chapter, Cai, Hwang, Jiang, and Silber (2015) discuss the progression of problem posing research along ten answered as well as 14 unanswered questions. Amongst others, they ask whether it is feasible to use problem posing as a measure of creativity, pointing at one possible connection between problem posing and creativity. There is, however, still much work to do in this field. Working in both the field of problem posing (Baumanns & Rott, in print) as well as in the field of mathematical creativity (Joklitschke, Rott, & Schindler, 2018), we were intrigued to examine the intersection of both fields (Fig. 1) as indicated by Silver (1994, 1997) or Cai et al. (2015). Ayllón, Gomez, and Ballesta-Claver (2016) conducted a review of this intersection. However, there are some uncertainties (details are explained below) in the content and it is not clear to what extent the review fully reflects the existing research literature. Therefore, this article presents an attempt at a systematic review of studies dealing with both problem posing and creativity published in highly ranked journals.

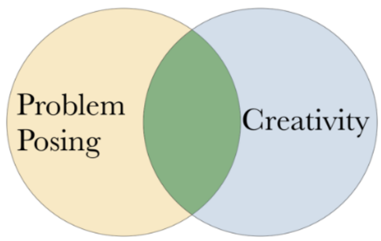

Fig. 1: Intersection of research on problem posing and creativity as focus of this paper

BACKGROUND

In the following, we provide a current theoretical understanding of mathematical problem posing, mathematical creativity, and their intersection.

Problem Posing

Problem posing has been emphasized as an important mathematical activity by many mathematicians (e.g., Hadamard, 1945; Cantor, 1966/1932) as well as mathematics educators (e.g., Brown & Walter, 1983; Silver, 1994; English, 1997). As an important companion of problem solving, problem posing can lead to flexible thinking, improve problem-solving skills, and sharpen learners' understanding of mathematical contents (English, 1997). There are two definitions of problem posing, at least one of which is used or referred to in the majority of research papers on the topic. The first definition was proposed by Silver (1994, p. 19), who describes problem posing as the activities of generating new problems and reformulating given problems. Both activities can occur *before*, *during*, or *after* a problem-solving process. The second definition comes from Stoyanova and Ellerton (1996, p. 518), who refer to problem posing as the "process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems". The authors also maintain a categorization for different types of problem-posing situations and differentiate between *free*, *semi-structured*, and *structured* problem-posing situations, depending on their degree of given information.

Creativity

Solving and posing complex problems often requires creative ideas; particularly in technology and science, this development is very important. Educational research also has an increased interest in research in this field (Singer et al., 2017; Joklitschke, Schindler, & Rott, 2018). Research on creativity goes back to at least the psychologist Guilford (1967) exploring the nature of intelligence. In his work, he differentiated convergent and divergent thinking abilities, the latter encompassing *fluency*, *flexibility*, *originality*, and *elaboration*. These dimensions are apparent in the well-known Torrance Tests of Creative Thinking (TTCT; Torrance, 1974), which is an attempt to make creativity measurable quantitatively. In the field of mathematics education, several researchers draw on this composition to assess mathematical creativity (e.g., Leikin & Lev, 2013; Pitta-Pantazi, 2017). Other researchers (e.g., Liljedahl, 2013) look at creativity using a model consisting of the phases *preparation*, *incubation*, *illumination*, and *verification* and thereby follow Hadamard (1945). In early research, mathematical creativity was attributed exclusively to experts (e.g., Hadamard, 1945) and was therefore an *absolute* characteristic. However, a number of researchers assume that creativity may also be attributed to students, their processes, or products and view creativity as a more *relative* construct (e.g., Leikin & Lev, 2013).

Intersection of problem posing and creativity

As we explained in the introduction, a considerable part of studies investigating the intersection of problem posing and creativity refers to the articles of Silver (1994 and 1997, resp.), which is why we highlight Silver's key statements in the following.

In 1994, Silver points out that various tests to identify creativity include problem-posing situations; thus, it is reasonable to assume a connection between problem posing and creativity. However, he states that the nature of this connection remains uncertain and needs further investigation. In 1997, Silver considers Torrance's (1974) categories of *fluency*, *flexibility*, and *originality* as key components of creativity and provides instructional suggestions how to foster creative activities in classrooms through problem posing. Furthermore, Silver (1997) emphasizes that "the connection to creativity lies [...] in the interplay between problem posing and problem solving" (ibid., p. 76). In the following, we focus on problem posing and its relation to creativity and describe the findings in this field on the basis of the following research questions: (1) *What kind of (and how many) journal articles exist dealing with the intersection of mathematical problem posing and mathematical creativity?* (2) *To what extent is this intersection conceptualized?*

METHODS

For this review, we used the preliminary work of two literature reviews on problem posing (Baumanns & Rott, in print) and on mathematical creativity (Joklitschke, Rott, & Schindler, 2018). We focused on (a) databases from the seven A*- and A-ranked journals (Törner & Arzarello, 2012) and on (b) the *Web of Science* within selected categories on mathematics and its education. In the databases from (a), we used the search term *problem posing* for all available years up to 2017. This procedure led to 332 articles. We then read all abstracts and extracted all articles that have *problem posing* either in their titles, abstracts, or keywords; this led to 48 articles. Furthermore, we consulted the database (b) *Web of Science* for the years 1945 to 2017. Excluding the already considered articles from the A*- and A-ranked journals, this led to another 81 articles. Within these 129 articles on problem posing from (a) and (b), we looked for the search term *creativ** within the titles, abstracts, and keywords to identify articles that potentially deal with the intersection of mathematical problem posing and mathematical creativity. In total, only 15 articles (eleven from the A*- and A-ranked journals and four from the Web of Science) remained.

In order to examine those 15 articles in a systematical and criteria-led way, each article was carefully read and assigned to one of the following genres: *theoretical contributions*, *review articles*, *perspective or opinion*, *empirical research with mainly qualitative methods*, and *empirical research with mainly quantitative methods*. Thereafter, the articles were examined with regard to their content. Due to our research question, we concentrated mainly on the conceptualizations of problem posing and creativity and the implementation of the empirical research if there is any. Thereby, the following questions were decisive: *What is the main message of the article? Which theories and conceptualizations are cited? How is the relation between problem posing and creativity represented?* Based on these questions, clusters were formed inductively to classify the articles into coherent groups that represent different approaches at the intersection of problem posing and creativity.

RESULTS

Introduction of the reviewed articles (Research question 1)

In the following, the 15 articles are presented and sorted by their genres.

Theoretical contributions: Two articles from our data set – the already mentioned articles by Silver (1994 and 1997, resp.) – were considered as theoretical contributions to the intersection of problem posing and creativity. Since the central focuses of the articles have already been covered above, we refer to the background for additional information.

Both articles are widely recognized as milestones in the (back then) young research fields on problem posing, mathematical creativity, and its intersection, respectively.

Review articles: One article written by Ayllón et al. (2016) is a review article summarizing central results regarding the relationship between creativity, problem posing, and problem solving. Apart from some inconsistencies and inaccuracies (e.g., wrongly assigned contents), the article clearly fits the topic of our review. However, as the number of articles considered to reflect the state of research of the intersection of problem posing and creativity is limited (three of four cited studies are discussed here as well; the fourth article does not meet our search criteria), the article by Ayllón et al. is not further considered.

Perspective or commentary: Two articles were categorized as commentary articles as they neither report on empirical studies, nor provide a theoretical discussion on problem posing or creativity. (1) Haylock (1997) presents examples of tasks designed to identify creativity in 11-12-year-old students. Highlighting *overcoming fixation* as a key component of creativity and referring to Guilford's and Torrance's ideas of *divergent production*, Haylock discusses specially designed problem-solving, problem-posing, and redefinition tasks that have been tested in previous studies. (2) Sriraman and Dickman (2017) discuss the use of *mathematical pathologies* (i.e. unpleasant counterexamples in the sense of Lakatos) to foster creativity in the classroom. In addition to historical examples of pathologies, the authors present examples from current classrooms in which students explore counterexamples or "incorrect" methods leading to correct results (e.g., a misinterpretation of rules to deal with fractions). Sriraman and Dickman then propose using the Lakatosian heuristic (conjecture – proof – refutation), to pose interesting problems that productively deal with pathologies and counterexamples.

Empirical research with mainly qualitative methods: In two articles, qualitative empirical studies are presented. (1) Leung (1997) correlates posed problems of 96 grade five students from Taiwan working on 18 different initial situations and describes those creatively posed problems in terms of change of content and context. (2) Voica and Singer (2013) investigate cognitive flexibility (i.e. variety, novelty, and change in framing) in problem posing. They analyze the products of 42 students with above average mathematical abilities working on structured problem posing situations.

Empirical research with mainly quantitative methods: In four articles, quantitative empirical studies are reported. (1) Bonotto (2013) investigates the potential of so-called *artifacts* (i.e. real-life objects like restaurant menus, advertisements, or TV guides) to stimulate critical and creative thinking. Additionally, she analyzes problem-posing and problem-solving products from 71 primary school students (stimulated by artifacts), using Guilford's categories of fluency, flexibility, and originality. (2) Singer, Voica, and Pelczer (2017) assess the cognitive flexibility (as an indicator for creativity) of 13 prospective teachers by analyzing the products from geometric, semi-structured problem-posing situations. (3) Van Harpen and Presmeg (2013) investigate the relationship between mathematical problem-posing abilities and mathematical content knowledge among high school students from three different countries. Similar to Bonotto (2013), they analyze the students' problem-posing products using the dimensions of fluency, flexibility, and originality to assess the students' creativity. (4) Van Harpen and Sriraman (2013) also use those dimensions to analyze problem-posing products of 218 high school students from the USA and China.

Excluded articles: Four articles (in alphabetical order) had to be excluded for reasons that are outlined below. (1) Ernest (2015) discusses social outcomes of learning mathematics in school by presenting standard aims as well as unintended (and often negative) outcomes (e.g., values, attitudes, and beliefs) and visionary aims (in which he emphasizes mathematical creativity through problem posing and solving) of school mathematics. Ernest uses neither the theoretical literature of problem posing research, nor that of research on creativity; he mentions those terms in his discussion and emphasizes their importance. (2) Patton (2002) uses biographical interviews to trace the recognition of creativity in the lives of famous entrepreneurs and scientists. To interpret his data, Patton uses the *systems theory view of creativity*, which suggests that creativity is not about being unique, but about being the first or being a "problem pioneer". He does cite literature from research on creativity (esp. Csikszentmihalyi), however, he does not use any literature from research on problem posing and, therefore, does not work in field of the intersection of both. (3) In the article by Poulos (2017), the author investigates the way an expert problem poser (a coach of the Greek team for the IMO) poses problems on the Olympiad level by describing two interviews. The author only uses literature from problem-posing research addressing experts' behavior. He does not cite any articles from creativity research and does not investigate the expert's creativity. (4) Singer, Sheffield, and Leikin (2017) wrote the introductory article to a ZDM special issue on creativity and giftedness in mathematics education. Thus, they do not present research results or new theoretical ideas in this article, but rather give a historical overview of research on those topics.

Cluster formation (Research question 2)

In order to summarize ideas and empirical implementations that can be found in research on a meta level, we will focus on the articles of the categories *empirical research* and *perspective or opinion*. Articles of the categories *theoretical contributions* and *review articles* will be held aside. The remaining eight articles can now be merged inductively into clusters. Some articles are assigned to multiple clusters.

Cluster I: Problem-posing situations to foster creativity: The first cluster contains articles that provide examples of problem-posing situations that are especially appropriate to foster creativity. For Cai et al. (2015, p. 17), the question which kind of problem-posing situations are appropriate to promote students' creativity is still unanswered. (1) Haylock (1997) presents tasks for his key components of creativity. For the component of *overcoming fixation*, he designed series of problems in which stereotypical approaches should be discarded. Additionally, he argues that in particular specially designed problem-solving, problem-posing, and redefinition tasks require the component of *divergent production*. (2) Bonotto (2013) investigates the potential of *artifacts* as semi-structured problem-posing situations to identify (see also II.a) and foster critical and creative thinking in the classroom. (3) Sriraman and Dickman (2017) use *mathematical pathologies*, the Lakatosian heuristic, and problem-posing activities to address students' creativity.

Cluster II: Identifying and investigating creativity through problem posing

II.a: Guilford's and Torrance's framework: This cluster contains all articles that – similar to Leikin and Lev's (2013) analyses of multiple solution tasks – use the categories *fluency*, *flexibility*, and *originality* based on works by Guilford (1967) and Torrance (1974) to investigate the participants' problem-posing products. (1) To measure *fluency*, Bonotto (2013) takes the total as well as the average number of problems created by the pupils working with her artifacts into account. To measure *flexibility*, the posed problems are categorized with regard to the number of details presented as well as the data introduced by the students. To measure *originality*, the rareness of the posed problems is considered: if a problem was posed by less than 10 % of the other pupils, it was considered original. (2) Van Harpen and Sriraman (2013) operationalize *fluency* and *originality* in the same way. *Flexibility* is measured by the total number of categories (e.g. analytical geometry, lengths, area, angles) the posed problems of a student can be assigned to. (3) Van Harpen and Presmeg (2013) use the same operationalizations of all categories as Van Harpen and Sriraman (2013). (4) The article by Leung (1997) could not be classified into this cluster quite as clearly; although Torrance's dimensions are mentioned and the test instrument is also based on the TTCT, these components do not play a role in the empirical part.

II.b: Other approaches to measure creativity through problem posing: The third cluster considers the approach by (1) Voica and Singer (2013), and (2) Singer, Voica, and Pelczer (2017). This cluster represents another line of research on mathematical creativity that is based on organizational-theory and discusses creativity in terms of cognitive flexibility (composed as cognitive variety, cognitive novelty, and change in framing) as an indicator for creativity. As a theoretical concept, they refer to the construct of cognitive flexibility to grasp the relationship between problem posing and creativity. As a methodological concept, they especially use the criteria of coherence and consistency of the posed problems. Additionally, Singer, Voica, and Pelczer (2017) consider the two dimensions of Geometric Nature (GN), and Conceptual Dispersion (CD). The GN assesses whether the posed problem is about finding sizes or specific computation (metric), or about geometric reasoning without computation (qualitative). The CD assesses if the posed problems are organized within clearly defined structures or systematically exploiting a configuration (structured), or the posed problems are e.g. disconnected from each other (entropic). They found that cognitive flexibility is inversely correlated to metric GN and structured CD.

CONCLUSION & OUTLOOK

In 1994, Silver stated that it is reasonable to assume a connection between mathematical problem posing and mathematical creativity; the concrete connection, however, was unknown. 25 years later, the gain in knowledge is still limited. In our literature review, we found only eleven articles in the whole databases of all A*- and A-ranked journals on mathematics education and on the Web of Science that address the intersection of problem posing and creativity. Since our review goes back to the founding dates of journals (e.g., ESM started in 1969, JRME in 1970, and FLM in 1980.), we realized that no articles addressing both problem posing and creativity have been published before 1994.

Analyzing the content of the articles under review, we were able to build coherent clusters focusing on (I) *problem-posing situations to foster creativity* and (II.a) *using problem posing to identify and investigate creativity via Guilford's and Torrance's framework* or (II.b) *other approaches to measure creativity through problem posing*. The inductively built clusters (I) and (II.a) can also be deductively sustained from Silver's theoretical considerations. In 1997, he focuses on fostering creativity through problem posing (and problem solving) which is the key aspect of cluster (I). Additionally, both of Silver's articles (1994, 1997) provide considerations to assess, identify, and investigate mathematical creativity through problem posing by applying Guilford's (1967) and Torrance's (1974) framework as in cluster (II.a). Interestingly, the assumption that problem posing can be used to measure creativity has not thoroughly been investigated, for example by correlation studies.

Apparent limitations of this review lie in the mere description of the articles central methodological and content-related elements. A more in-depth discussion about the chosen theoretical foundations, the research approaches as well as the results cannot be carried out at this point. Furthermore, the aim of this article was to look at the intersection of creativity and problem posing, which is why we did not consider other constructs such as problem solving or giftedness.

In order to expand the dataset of the existing articles, it would be of interest for a future review to also consider the databases of B-Journals (Törner & Arzarello, 2012), seminal collections on problem posing such as *Mathematical Problem Posing* (Singer, Ellerton & Cai, 2015), as well as the papers of the International Group for the Psychology of Mathematics Education (PME) and the International Group for Mathematical Creativity and Giftedness (MCG). Furthermore, additional keywords such as *innovat**, *invent**, and *divergent think** regarding creativity (cf. Joklitschke, Rott, & Schindler, 2018) or *task design*, and *problem formulation* regarding problem posing would extend the range of articles considered in the field of mathematical creativity. This consideration may lead to a wider data base, further clusters, and a better chance of comparing the different research approaches.

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PROVIDING FOR GIFTED LEARNERS IN THE REGULAR MATHEMATICS CLASSROOM: NEEDS AND THE WAY FORWARD

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Abstract: *In line with South Africa's constitution, the National Policy on Education (White Paper 6) clearly states for inclusive education to be a reality, the educational system should accommodate the learning needs of diverse learners, including those that are gifted. Nevertheless, more than a decade following the introduction of Education White Paper 6, most learners with giftedness who attend mathematics in regular classrooms are still having learning problems. There is no consensus regarding who should and should not be classified as a gifted learner in South Africa. The difference in opinion causes confusion in terms of identifying and supporting this group of learners. This paper examines the types of gifted learners and their needs. It further gives recommendations on how to meet the needs of different profiles of gifted learners.*

Key words: gifted learners, mathematics, inclusive education, profiles of the gifted and talented.

INTRODUCTION

Like any other developing country, the progress of South Africa's economy is also relying on its mathematically gifted learners. Sadly, these are one group of special learners that have been neglected for a long time. These are learners who their educational needs cannot be satisfied in the regular classroom because of their unique features and endowment. For a very long time the assumption has been that leaners in a regular classroom can be taught using the same methods. However, research (Cooper, 2009; Hertberg-Davis, 2009; Rogers, 1993) has shown that the "one size fits all" does not work because it assumes all learners learn in the same way.

Unfortunately many public schools in South Africa default to the traditional one size fits all because they are often overcrowded (Mncube, 2009; Modisaotsile, 2012) and do not have qualified teachers and resources to identify gifted learners and differentiate curriculum to suit the needs of such learners (Mhlolo & Marumo, 2017). As a result, gifted learners in mathematics are forced to attend in the regular classroom and learn the same content at the same pace with other non-gifted learners. This situation cannot change unless teachers understand the different types of gifted learners in their classrooms and how to meet their needs. The aim of this paper was to examine the types of gifted learners and their needs and it further gives recommendations on how to meet the needs of different profiles of gifted learners.

INCLUSIVE EDUCATION IN SOUTH AFRICA

The implementation of inclusive education in South Africa was instituted during curriculum changes by the government led by the African National Conference (Naicker, 2005). Transformation of the South African society with the initiation of inclusive education was in line with Salamanca Statement of 1994 (Engelbrecht, 2006). To ensure the implementation, the government drafted policies as indicated in the White Paper 6 (Department of Education, 2001). The implementation process of inclusive education in South Africa can be summarised as follows:

The White paper on Education and Training in a Democratic South Africa (1955); the South African Schools Act 84 of 1996; The White Paper on an Integrated National Disability Strategy (INDS) (1997); the National Commission Special Needs and Training (1997); the National Committee on Education Support Services (1997); the Education White Paper 6 Special Needs Education: building an inclusive education and training system (2011); the Draft National Disability Policy Framework (2008). Guidelines for the implementation of National Disability Framework (2008) and The United Nations on rights of Persons with Disabilities (2006) ratified by South Africa in 2007.

White Paper 6 (Department of Education, 2001) clearly states that for inclusive education to be a reality, there ought to be a conceptual change concerning the provision of support for learners that encounter difficulties in learning. The strategy that the Department of Education has adopted aimed at steering the implementation of policies concerning inclusive education. This policy has two major elements which are clarified in two sets of guidelines:

The National Strategy on screening, Identification, Assessment and Support (SIAS) guides inclusive education policy by describing how teachers should identify and assess learners in schools (Department of Education, 2008). The SIAS strategy offers guidelines to schools on early identification and support required by learners and also the guidelines on the central role of teachers in implementing the strategy (Dalton, Mckenzie, & Kahonde, 2012).

The Guidelines for Responding to Learner Diversity in the Classroom through Curriculum and Assessment Policy offer practical guidelines to education officials and teachers on planning to meet the needs of a diverse range of learners in classrooms (Department of Education, 2011). This document has been reviewed to include curriculum adjustments in the Curriculum and Assessment Policy Statement (CAPS) and the up-dated serves as a part of the CAPS (Dalton et al., 2012) orientation programme for school managers and teachers around the country.

Regardless of the enabling policies outlined above, it seems that in South Africa, implementing inclusive education has challenges (Dalton et al., 2012; Wildeman & Nomdo, 2007), and the difficulties arise from number of reasons such as lack of teachers' training and workshops, and issues that affect the education system at large including the role of special schools. Even though some teachers are found to understand inclusive education, other teachers do not have the knowledge and understanding thereof (Mayaba, 2010). Moreover, the new constitution emphasizes on the recognition of diversity. This implies an inclusive approach to education where all learners are entitled to an environment that is inclusive and supportive.

TYPES OF GIFTED LEARNERS

After several years of observations and reviews, Betts and Neihart (1988) identified six different profiles of gifted learners. The profiles of gifted learners can provide information for teachers about behaviour and need of gifted learners in mathematics classrooms. Profiles of gifted can also be used as the information for inservice and courses concerning the nature and needs of the gifted.

Type 1 – The Successful

The successful gifted learners are motivated achievers and perfectionistic. However their motivation may be directed mainly towards teacher acceptance rather than the full development of their abilities. Type 1 learners many believe will it make on their own, though they often get bored with school and use the system to get with as little effort as

possible. If not challenged, these type of learners may not learn the skills and attitude they need for future creativity and autonomy.

Type 1 learners need activities that push them out of comfort zone. Teachers need to give them more time to be with their intellectual peers as well as for personal curriculum.

Type 2 – The Creative

The challenging learners are divergently gifted, stand up for their convictions, challenge teachers in front of the class, and question rules and policies. Many schools fail to identify Type 2 learners and may feel frustrated because the school has system has not affirmed their abilities. If the creative learners are not challenged, they can exhibit inconsistent work habits and boredom. These learners may be "at-risk" of dropping out of school if their abilities are not supported.

Type 2 learners need to be placed with appropriate teachers who can direct instruction in interpersonal skills and reward new thinking.

Type 3 – The Underground

Type 3 gifted are generally middle school female learners, although males learners may also go underground as to hide their giftedness. The underground learners may start as successful but later deny their abilities in order to feel more accepted in a non-gifted peer group. Their needs are often in conflicting with teachers' expectations.

While Type 3 are undergoing transition, teachers should not allow them to abandon classes and projects. Alternatives for meeting their academic needs should be explored. They need welcoming environments to develop self-awareness and self-acceptance.

Type 4 – The At-Risk

The at-risk learners show resentment, depression and anger because the school system has not provided for their needs and feel rejected as a result. Type 4 learners may show anger by withdrawing or acting out and responding defensively. This type of learners are often viewed by other as average or below average and are not motivated for teacher driven rewards. They are frequently gifted learners who were identified very late, and as a result their self-esteem is very low.

Teachers need to work closely with an adult the learners can trust. Type 4 learners need individual counselling, out-of-classroom learning experiences and non-traditional study skills.

Type 5 – Twice/Multi Exceptional

The Multi Exceptional learners are emotionally or physically challenged, or have learning disabilities. Teachers and peers may have difficulties recognising their abilities, instead they tend to focus on the weaknesses. Such learners can feel rejected, powerless, isolated or frustrated. They may have disruptive behaviours or untidy handwriting leading to inconsistent school work.

Type 5 learners need the supportive teachers and peers who will remind them about their giftedness. They may further benefit from acceleration in area of strengths and provided with opportunities for exploration and investigation.

Type 6 – The Autonomous Learner

The autonomous learners demonstrate some Successful characteristics and create opportunities for themselves. They are well motivated, self-directed, confident, and are able to take appropriate academic risks. Teachers and peers give them positive attention and consider them responsible.

Autonomous learner needs opportunities that promote development of a long term plan study and if possible, removal of time and space restrictions for their studies.

THE WAY FORWARD.

The following are recommendations to cater for the needs of gifted learners in the regular classrooms:

i) There is the need to train both preservice and inservice teachers in the area of gifted education.

ii) There is the need to design programmes that cater for the needs of gifted learners and be constantly monitored and evaluated.

iii) There is the need for proper funding for gifted education.

iv) There is the need to be aware of strengths and weaknesses of learners in mathematics classrooms as gifted learners may not excel in all areas and they may be ahead or behind other learners.

v) There is the need to establish inclusive classrooms by providing an environment which allows all learners to advance at their own rate of learning.

vi) There is no need to put more pressure and unrealistic expectations on gifted learners.

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SETTING THE CEILING TOO LOW FOR MATHEMATICALLY GIFTED STUDENTS IN SOUTH AFRICAN SCHOOLS

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Abstract: *In in clusive classrooms empirical evidence shows that half of gifted and talented students do not perform to their best abilities because the assessment tasks were insufficiently difficult to measure students' true ability or knowledge. This theoretical paper uses the concept of frame to analyse the different aspects of the South African education system that contribute to setting a low ceiling for mathematically gifted students. This is important because setting low ceilings tends to promote gifted underachievement as tasks are too easy and do not encourage challenge or sustained effort. Although many frames could be used to analyse the South African education system, in this paper I argue that (a) the way inclusive education is conceptualised and implemented, (b) the way the curriculum is structured, (c) the level at which the pass mark is pegged as well as (d) the teacher competencies can all lower the ceiling for gifted students thereby inhibiting the maximization of their potential.*

Key words: gifted students, underachievement, inclusive education, frame, low-ceiling

INTRODUCTION

Within the economic development debates, the mathematically gifted learners have been described as "the world's ultimate capital asset" as they guarantee a constant reservoir of individuals who will lead, both research and development toward a knowledge-based economy. This suggests that education systems have a special mandate to nurture their mathematically gifted students if they haven to remain competitive in a 21st knowledge-based economy (KBE). However, in post-apartheid South Africa education, reforms seem to have lost sight of this important mandate because stakeholders have been hostile to and resentful of gifted education programs which are critical for academic talent development (Kokot, 2010).

STATEMENT OF THE PROBLEM

South Africa does not have special provision for gifted education hence gifted students are in the regular classroom following a decision guided by the inclusive education philosophy adopted at attaining democracy in 1994. In 'inclusive' classrooms empirical evidence shows that half of gifted and talented students do not perform to their best abilities. To some, these assertions may appear confusing, because bright students appear to be performing well if we look at the standard performance evaluations. However, in a standards-driven environment, inclusive classrooms focus on bringing all students to a certain set level of proficiency, which is low according to Winner, (1996). The implication of meeting this low proficiency level is that many of our brightest students are being denied the opportunity to learn at appropriate levels (Colangelo, Assouline & Gross, 2004). Consequently, gifted students find school playing little or no role in the development of their gifts because inclusive schools have low standards that lead to their underachievement. The South African school system can be described as one such example pursuing low standards for gifted students in the sense that all students are expected to attain a certain minimum level of knowledge and competence. It is against this background that the South Africa education system provides an ideal example to study the dangers of uniformed and undifferentiated equity thinking in educational settings in relation to intellectually gifted individuals.

PURPOSE STATEMENT

This paper analyses the different aspects of the South African education system that contribute to setting a low ceiling for mathematically gifted students. This is important because setting low ceilings tends to promote gifted underachievement as tasks are too easy and do not encourage challenge or sustained effort. When this happens, the price we pay is "lost academic growth, lost creative potential and sometimes dropping out or lost enthusiasm for educational success and eventual professional achievements and substantial contributions to society" (Davis & Rimm, 2004). In this paper giftedness is defined in accordance with Gagné (2010) whose definition was considered appropriate for the South African inclusive education system. According to Gagné (2015) *giftedness* designates the possession and use of untrained and spontaneously expressed superior natural abilities (called aptitudes or gifts), in at least one ability domain, to a degree that places an individual at least among the top 10% of his or her age peers. In this regard Gagné argued that the 1:10 or the top 3 achievers in a regular class of 30 already distance themselves very significantly in terms of ease and speed of learning. He referred to such learners as mildly gifted or the 'garden variety'— a common English expression in the USA that means the 'most common group'. In education systems that are guided by the inclusive philosophy, these 'mildly' gifted learners spend most of their time in regular classrooms hence it can be argued that every teacher should be regarded as a teacher of at least the mildly gifted. This conceptualisation then allowed me to focus my analysis on the general school system since (a) South Africa does not have special educational provisions for gifted students and (b) every teacher should be regarded as a teacher for gifted learners. Many studies of egalitarian and inclusive education systems have identified factors that can inhibit/support the development of gifted students within such systems (e.g. Winner, 1996; Persson, 2010)

FRAMES FOR ANALYZING GIFTED EDUCATION IN SOUTH AFRICA

In order to give structure to my analysis of education in South Africa, it is helpful for the reader to understand the framework that I used for that analysis. The conceptual tool of 'frame' served as a guide for the analysis of the South African school system. A frame may be defined as a psychological construct that furnishes one with a prevailing point of view that manipulates prominence and relevance in order to influence thinking and, if need be, subsequent judgment as well (Wendland, 2010). Although there are many frames from which the South African education system could be analyzed, in this theoretical paper I argue that (a) the way inclusive education is conceptualized and implemented, (b) the way the curriculum is structured, (c) the level at which the pass mark is pegged as well as (d) the teachers' competencies can all lower the ceiling for gifted students thereby inhibiting the maximization of their potential.

RESEARCH QUESTIONS

Consistent with the frames that shaped this paper four research questions were raised as follows:

1. How is inclusive education conceptualized and implemented in the South African education system?
2. To what extent does the South African curriculum allow/inhibit gifted students to develop to their full potential?
3. What is the pass mark at exit level in mathematics and to what extent is this level stretching gifted students to reach their full potential?
4. To what extent are teachers competent to develop gifted students to their full potential?

CONCEPTUALISATION AND IMPLEMENTATION OF INCLUSIVE EDUCATION

The way in which countries realize inclusion within their classrooms can take different forms which can contribute to either lowering the bar or setting the bar high for gifted students. Historically there have been two main approaches to inclusive education which have been associated with the inclusion and or exclusion of gifted students. One approach focuses on what is called ‘equality of opportunity’ while the second view is more concerned with quality of that access or excellence in education. Achieving equality of opportunity requires only that access to education be reasonably distributed. At its worst, an ‘equality of opportunity’ perspective could be and was held to justify school provision that was segregated based on ethnicity or religion. In this view, it is access to education that was critical, and the responsibility of the state was the provision of opportunities to participate; whether people choose to take advantage of that access or are successful in doing so was not a primary focus of inclusion. Unfortunately, this has been the dominant view in many formally colonized African states and in the case of South Africa, empirical evidence has shown that the country has taken too much of this equity approach at the expense of an excellence approach (OECD, 2013). Oswald & De Villiers (2013) warned of the negative effects that an egalitarian and equalizing approach to education had in the field of gifted education. This egalitarian approach as well as the overloading of teachers in over-crowded classrooms, left the gifted learner with minimal attention in such classrooms. This has resulted in current practices in South Africa where current educational policies and practices have focused on lower levels of mathematical capabilities where teaching is based on a criterion of averages and achievements have always been measured from a social equity perspective. This is a typical example of lowering the bar for gifted students which does not help such students to develop to their full potential.

CURRICULUM STRUCTURE

The way a curriculum is structured can also contribute to lowering the bar for gifted students. The curriculum is the central means for enacting the principles of inclusion and equity within an education system. The South African curriculum is called the Curriculum and Assessment Policy Statement (CAPS) which is developed and monitored by the national Department of Basic Education’s (DBE). CAPS is premised on the view that inclusive education would provide "a cornerstone of an integrated and caring society" and the regular classroom would meet the needs of diverse learners. However, if learning in inclusive classrooms is defined narrowly as the acquisition of knowledge presented by a teacher, schools will likely be locked into rigidly organized curricula and teaching practices. South Africa is a typical example where learning has been narrowly defined leading to a single curriculum for all learners learning in mainstream classrooms. Practice shows that learners who are not doing well and the average are the ones that most curriculums are designed for, leaving out the gifted in the current classroom designs and practices. In terms of the CAPS document, which is based on inclusive education policies, critics have warned that the 'one size fits all' trends in South African educational provisioning and curriculum development, although intended to ensure more equitable access to education for the poor, in many instances effectively exclude the very children it was intended for (Harley & Wedekind, 2004).

PASS MARK

Pass marks are a typical example where the bar can be set either low or high for gifted students. For example, in South Africa teachers are heavily unionized and because of that strength they have fought vehemently to lower the bars where learners need a mere 30% to pass Mathematics at metric level. Setting such a low bar does not allow gifted children to reach their fullest potential in terms of cognitive, emotional and creative capacities. For example, in a similar study by Persson (2010) gifted students commented that school was too easy but the price they had to pay for being high achievers was so punitive that they had to shut down their brain to adapt to the teachers' level of instruction. It was hell as one gifted student commented; "If I did my math too quickly, the teacher made me erase everything and start all over again so that I could finish with the rest." So "I deliberately kept my achievement low so I would fit in."

TEACHER CONTENT KNOWLEDGE

The level of teacher content knowledge can also be a contributory factor to setting a low bar for gifted students. Gifted students can only develop to their full potential if they are taught by properly trained and highly competent teachers. For example, in a South African study by Mhlolo (2017), despite being creative in solving an algebra problem, a student was reprimanded by the teacher for not following instructions and that if it was in the exam, she was not going to get credit. In a similar study by Persson (2010) some gifted participants were perceived as being a threat to their teacher. One participant commented; "My teachers felt threatened by me. It always felt like the teachers were bothered by me" and "I always sought extra work in school but was met by a formidable resistance". So instead of being given exciting learning assignments I was made to help others and my teachers never allowed me to proceed and progress my own learning." In the South African schools, the teaching of mathematics is amongst the worst in the world. For the past 20 years, research has shown that South African teachers, especially mathematics teachers, have inadequate subject content knowledge. Taylor & Taylor (2013), in a report of the Department of Basic Education's National Education Evaluation and Development Unit (NEEDU), argue that poor learner performance in most schools is largely due to the poor subject knowledge of teachers, especially in mathematics.

CONCLUSION

This paper raised four questions one each around the implementation of inclusive education, the structure of the curriculum, the level of pass marks in mathematics and the competencies of teachers. In all four areas I have shown how the bars have been set so low in South Africa that gifted students find it too easy to jump those hurdles. A conclusion that can be drawn thereof is that the education system is not maximizing the potential of its gifted students. This confirms the National Planning Commission (NPC) (2011) report which acknowledged that gifted learners were a critical component of the country's capability which the education system has neglected for decades.

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WHAT IS EPISTEMOLOGY OF THE IMAGINATION? THEORY-EPISTEMOLOGICAL BASES TO MATHEMATICAL REASONING

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***Abstract**: Given the need to design new strategies for the development of mathematical reasoning to overcome the mathematical reasoning and creativity inhibition, the epistemology of the imagination is presented as a current option that consider the symbolic-imaginative reasoning of the cognitive triad as a fundamental concept. Through the psycho-sociogenic method, the work presents some theoretical and epistemological bases that could help teachers in the development of new learning strategies.*

Key words: Epistemology of the imagination, mathematical reasoning, symbolic-imaginative reasoning, psycho-sociogenic method, cognitive triad.

INTRODUCTION

The epistemology of the imagination was proposed more than a decade ago by Rodríguez-Salazar (2015; 2016), based on Piaget's approach without exhausting itself in it, but rather, he goes beyond. From this position epistemology is understood as a reflection that emanates from the scientists themselves and no longer as a philosophical speculation. This indicates its autonomy as a science different from philosophy of science.

In this framework, mathematical reasoning and ability appears today as a prior capacity to strengthen mathematical creativity in children (Huang, et. al., 2017; Stolte, Kroesbergen & Van Luit, 2018; Schoevers, et. al., 2019). Stolte, Kroesbergen & Van Luit (2018), argue that mathematical creativity "is commonly operationalized as divergent thinking" composed by fluency, flexibility and originality. While mathematical ability is an essential prerequisite for mathematical creativity. Therefore, it is needed new strategies that link both.

Consequently, a neo-Piagetian theory could help to a better understanding of the cognitive processes in mathematical creativity and giftedness (Guénole, et. al., 2015; Kivkovich, 2015). Therefore, it is necessary to propose from a current epistemological approach the theoretical bases to elaborate programs and strategies of action. It is in this sense the epistemology of the imagination allows us to explore ways by emphasizing the symbolic component.

The present work approaches the fundamental expositions of the epistemology of the imagination at first, to lay the foundations that will allow outlining the relationship between the symbolic-imaginative and the mathematical reasoning. In the third section, some arguments are offered to configure theoretical bases about the game and mathematical reasoning. Finally, a brief open conclusion of the total work is presented.

EPISTEMOLOGY OF THE IMAGINATION

The epistemology of imagination argues the bond that unites the evoke actions with the material actions; it is the symbolic-imaginative experience that makes possible the creation of possible realities on how to solve the tangible reality problems. Thus, the subject who observes the operation of some artefact (material experience) goes on to relate it to his knowledge and experiences background, creating images or scenarios of materially possible realities (symbolic-imaginative experience), to subsequently design some product that fulfills a specific function (formal experience).

However, it is legitimate to ask: how is the concept of imagination understood under this approach? Well, this is addressed to the Kantian idea of pure forms of understanding, where imagination is a free scheme of experience that, on the one hand, is conceived as a rational thought and, on the other, plays a mediating role between intuitions and concepts (Rodríguez-Salazar, 2016: 82-83).

In the same Kantian line, the imagination is productive and a faculty that synthesizes knowledge *a priori*. In this way, imagination is no longer the product of fantasy or metaphorical and philosophical speculation of reality, but the apex of knowledge that configures possible realities in cognition to be returned to the material in the form of solution.

This way of understanding the imagination is still limited, so Epistemology of Imagination (Rodríguez-Salazar 2015; 2016: 89) takes Piaget's stages of psychogenic development theory and the three sets of actions that illustrate the subject-object relation, which are: 1) materials, referring to the act of the subject on the objects of reality; 2) evoked, mental prolongations of the material actions that shape reality; and, 3) operative, organization of external reality through formal representations.

This proposal assumes an extended notion of experience to integrate the symbolic sphere in this relation, therefore it sustains "that the three sets of actions coexist in every subject and continue to function coordinated throughout life, forming a general structure of cognitive behavior, that is, a cognitive-behavioral structure" (Rodríguez-Salazar, 2015: 164). This new form of relationship can be expressed in the following figure:

$$\frac{\boldsymbol{MOA}}{\boldsymbol{MMA}}\boldsymbol{MEA} \leftrightarrow \frac{\boldsymbol{A}}{\boldsymbol{R}} \leftrightarrow \boldsymbol{ICPR}\frac{\boldsymbol{ICFE}}{\boldsymbol{IPR}}$$

FIGURE 1. Rodríguez-Salazar's proposal of the Subject-Object relationship under the expanded notion of experience. Source: Adaptation of the figure presented in Rodríguez-Salazar (2015: 165)

Where the abbreviations are: MOA = Mechanism of Operative Actions; MMA = Mechanisms of Material Actions; MEA = Mechanisms of Evoked Action; A = Actions; R = Reality; ICPR = Imaginary Configurations of Possible Realities; ICFE = Images Configurations of Formal Structures; IPR = Intrinsic Properties of Reality. Therefore, the Figure 1 represents an extended subject-object relationship model, where the actions mechanisms of the subject, impact on reality producing a set of cognitive configurations to structure reality. Thus, when some problem is presented to the subject, the cognitive configurations produce some possible realities to solve it. Hence, those imaginary scenarios address the actions mechanisms of subject to test them on reality to re-structure reality. This produce a new set of cognitive configurations and this process goes on.

However, the evoked actions have special epistemological value that consists in "creating the imaginary configurations of materially possible realities" (p. 167). Therefore, it is interpreted in this work that imagination is the bridge between the mental and the social spheres; the individual subject arrives at the social world. Therefore, through these mental configurations, the subject organizes a socially accepted reality, which implies that subjectivity is objectified through said organization. In this sense, it is observed that the epistemology of the imagination offers a model to understand social reality through the actions with which the subject structures its reality.

Rodríguez-Salazar (2016: 89) argues that this structuring of the subject "Taken to the social field and within the framework of genetic epistemology, Piaget establishes a parallelism between the structures of practical intelligence and formal operations, with the structures of social groups."

In this way, Rodríguez-Salazar & Rosas-Colín (2011) also argue from a neo-Piagetian position that the form and function of thought are social nature while the content of thought is individual. In this work, it is argued that although the social factor is not uni-determining for the structure and function of thought, it acquires a co-evolutionary character with the cognition that has the symbol-imaginative structures as its intermediary.

SYMBOLIC THOUGHT

According to Piaget, the psychogenic explanation oscillates between the physiological and logical spheres, consisting in explaining how it is possible and in what way the construction of operational stages are carried out. Which means that this type of explanation is based on knowing the cognitive evolutionary process of the subject in the overcoming of initial to terminal stages?

Piaget (1950/1994) proposes two general stages of cognitive development. The first is the sensory-motor, where structures are based on the satisfaction of needs. The second is the operative, which is composed of three moments: 1) pre-operative [although this is considered more as a link between first and second stages], consisting of the acquisition of language and in the first configurations of intentional actions without being fully structured; 2) Concrete operations, where the subject establishes the direct relationship with the environment and regulates this relationship based on mental schemes of action; and 3) formal operations, where the subject can achieve the abstraction of reality by developing a hypothetical-deductive thought.

Then the psychological fact consists of three inseparable aspects: 1) structure of behavior (normative-cognitive aspect); 2) economy or energy (affective aspect [values]); and, 3) symbol systems (signifiers of the operative structures). Therefore, it is observed that the psychological fact can be studied from the structuring aspects of reality, which leads to determine what the values of the subject are; it means what are those things that cause pleasure or displeasure and observe them later with its structures and asses the meanings that subject assigns to objects.

In that sense, on Piaget's lessons of *The psychology of intelligence*, first published in 1947, argues that there is much work to be done "between preverbal intelligence and operative thinking so that reflexive groupings are constituted, and if there is functional continuity between the extremes, it is indispensable to construct a series of intermediate structures in multiple and heterogeneous levels" (1947/2013: 156).

Derived from the semiotic function (Piaget & Inhelder, 1969/2000: 64) that appears at the end of the sensory-motor stage, the subject configures a set of operations that are divided into two kinds of instruments: the symbol and the sign. Piaget (1947/2013: 160-161) establishes that all kinds of motor or cognitive activity is symbolic insofar as it consists of relating a signifier to a signified reality, whereas the sign consists of arbitrary conventions about reality; that is, the signs, as their conceptual unity, are social while the symbol is individual. However, any symbol can be collective as it is socialized with a group, so it is configured as half symbol and half sign. In this way, the mathematical language corresponds to the signs while mathematical reasoning constitutes a kind of collective symbolic configurations.

When subject acquires language the symbolic schemes or actions schemes that evoke absent situations begin; these schemes appear in the child's game, and they contribute to the understanding of signs through language, which can be defined as a *general symbolic function*.

The child in his early years and during the development of language can strengthen what Piaget calls as egocentric assimilation of reality to structure reality to his own interests. In other words, as long as the child strengthens the symbolic game at this stage to model the images of reality by the self, his structures will have greater capacity to decentralize and, therefore, to formalize those images.

According to the epistemology of the imagination (Rodríguez-Salazar & Rosas- Colín, 2011), there is a cognitive triad that, through the coordination of its spheres, makes possible the structuring of reality. This triad consists of Practical Reasoning (PR), Symbolic-imaginative Reasoning (SIR) and Formal Reasoning (FR).

In this way, both the PR and the FR converge and communicate through the SIR. In fact, it is in this sphere that it serves as a link between the sensitive experience and the evocation of structured images in ways to scheme operations of action on that reality.

GAME & SYMBOLIC ASPECTS OF MATHEMATICAL REASONING

Stolte, Kroesbergen & Van Luit (2018), argue that mathematical creativity "is commonly operationalized as divergent thinking" composed by fluency, flexibility and originality. While mathematical ability is an essential prerequisite for mathematical creativity. In their paper they argue that one factor of mathematical creativity base could be inhibition. If inhibition is reducing, the relation between mathematical ability and originality could be strengthened. However, the paper approach is based on cognitive sphere. Hence it is necessary to go deeper and include the symbolic and social sphere in order to elaborate new strategies that link mathematical creativity and ability.

Today's strategies to develop the mathematical reasoning have been addressed to involve all the children social spheres (family, friends, educators, etc.) (Kivkovich; 2015; Trifu, Trifu & Trifu, 2016). Also, it is known that "children do not possess a true concept of number until they are able to reason on numerical quantity" (Viarouge, Houdé & Borst, 2019). Therefore, it is considered in this work the construction games are vital for the development of reasoning in general and mathematical reasoning in particular.

Thus, through activities at classroom and home dedicated to strength the configuration of images of reality in response to the emotional and intellectual needs of the child, it can contribute to constitute a solid base that will later be developed by the socialization in the rules games.

Cognitive or motor action involves a symbolic component, so the child needs to adapt to in the world of adults to find an emotional balance. As Piaget says:

> Obliged to adapt incessantly to a social world of elders [...] It is, therefore, indispensable to its affective and intellectual equilibrium that may have a sector of activity whose motivation is not adaptation to the real, but, on the contrary, the assimilation of the real to the self, without coercion or sanctions. (Piaget & Inhelder, 1969/2000: 65)

In this way, the symbolic game appears as a way to transform the real in terms of the needs of the self. In other words, through this kind of play, the child, in his egocentric sphere, assimilates his environment to his own interests. Then, to accommodate these assimilations according to what is presented as external. Finally, in its decentralization process, the child achieves a balance between both processes.

Language has an important role in achieving such a process. Therefore, it is essential that the children have their own means of expression that allows them to build a system of signifiers according to their interests. This is how a symbolic language forms and can be modified according to their needs.

Mathematical reasoning can be addressed in this sense. As long as math is symbolic conceived its teaching strategies must involve the children symbolic sphere. According to Piaget & Inhelder (1969/2000: 66) there are four types of games that are involved in the development of the subject: 1) Exercise game, which consists in repeating an activity for pleasure; 2) The symbolic game, which is based on evoked actions; 3) Games of rules, which is the social expression of the child; and, 4) construction games, which is a link between the second and the third, whose main function is to offer a base for adaptation or problem solutions.

For this paper nature, there is not space to get deeper in the explanation of those statements. However, the thesis of construction games as strategy to link the social and symbolic components to strengthen the mathematical reasoning and creativity is presented.

FINAL ARGUMENT

As a conclusion, it can be argued that with the coordination and communication of the spheres that make up the cognitive triad and with the mechanisms of actions (figure 1), schemes and structures strengthened can be developed in the child, also extended in the symbolic experience, to achieve the formalization of said images in mathematical signs. If the teacher practice takes into consideration that the development of symbolic thought occurs through the development of sensory-motor structures configured in images that are then brought into evoked actions forms, strategies can be organized that strengthen these assimilations considering the intelligence evolutional development in the child. The statement above could be the teaching principle of mathematical reasoning and creativity. This proposal encourage the construction game to overcome the inhibition in mathematical reasoning, using the symbolic as a link between, not just mathematical creativity and ability (Stolte, Kroesbergen & Van-Luit, 2018), but social and cognitive spheres too.

However, there is too much to be done in the symbolic field of cognitive development of human beings. The theoretical systematization of mathematical reasoning thought symbolic sphere can be a first step to move towards other dimensions of reasoning. In this way, this work presents one of the arguments proposed by the *Novo Cimento* research group, which aims to propose an epistemological alternative in the study of knowledge and its mechanisms in the subject and its relationship with the environment. Therefore, it is the collective effort that develops the ideas of an epistemology that takes a position from scientific reflection to situate itself at east of traditional paradigm and face the challenges of the 21st century (Rodríguez-Salazar & Díaz-Barriga, 2018).

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RESEARCH

GIRLS' PERFORMANCE IN THE KANGAROO CONTEST

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Abstract. *The issue of attracting girls to mathematics has captured our attention when we were analyzing data from the final stage of the Kangaroo mathematics contest in Israel. With general finding showing boys having better results, further analysis of differences across Grades 2-6 indicates that in some grades the gap is smaller than in others. For instance, only insignificant differences were found in Grades 3- 4 for all difficulty levels. Furthermore, on some tasks, the girls' performance was better than the boys'. In this respect, continuous investigation is needed to examine possible factors that make it happen. Furthermore, qualitative data could be collected and analyzed about young students' thinking when solving different tasks to uncover other possible hidden factors that influence mathematical performance by girls in Kangaroo contest.*

Key words: gender, spatial ability, mathematics performance, competitions

PROBLEM STATEMENT AND CONTEXT

Many educators express concern regarding the gender gap in mathematics performance and the underrepresentation of women in science, technology, engineering and mathematics (STEM) careers (Hyde et al., 2008). Gender inequity is particularly evident in data related to the number of girls that participated in the International Math Olympiad, or the number of female professors in university mathematics and engineering departments (Hyde & Mertz, 2009).

Several researchers pointed at mathematics performance in favor of boys (Aunola et al., 2004, Githua & Mwangi, 2003, Marsh et al., 2008), whereas others (Lindberg et al., 2010) claimed that no significant gender gap exists in mathematics. Moreover, Robinson and Lubenski (2011), and Brown and Kanyongo (2010) showed that over the last four decades, girls have achieved slightly better grades in mathematics than boys have.

As Halpern et al. (2007) pointed out, that influence of gender as well as other factors such as early experience, educational policy, and cultural context are unclear and has to be considered in systematic investigations. Gherasim et al. (2013) also argued that there is a need for more studies on gender differences in order to fill the gaps regarding the mechanisms that are conducive to enhancing mathematical performance.

In what way do gender differences appear (if at all) in the context of mathematics competition? Indeed, Niederle and Vesterlund (2010) found that gender difference in competitive performance does not reflect the differences in non-competitive performance. Gneezy et al. (2003) even revealed that gender gap in performance under competition conditions is three times greater than in non-competitive environments. Leedy et al. (2003) studied beliefs held by students participating in regional math competitions, as well as those held by their parents and teachers. They found that mathematics is still viewed as a male-dominated discipline, while girls and women fail to acknowledge the existence of the bias. They argue that the task of the school is not to ignore or deny differences in learning styles, attitudes and performance but to acknowledge and use them for developing strategies aimed at providing gender-equitable education.

However, there is not enough data about how gender-related differences are manifested in mathematics competitions and what patterns emerge from these differences.

We started investigating gender issues in the context of the Virtual Mathematical Marathon by studying participation and performance (Applebaum et al., 2013). While observing students' participation during the first two years of the competition, we found that girls and boys showed similar patterns regarding the decision to remain in the competition, or to abandon it, regardless of the results in previous rounds.

In this paper, we analyze gender differences based on data from the 2016 Israeli competition, as part of an International Kangaroo Contest, with the winners determined separately in each participating country. In the first stage, students took the test administered on the internet (students had to identify themselves) from home. Those who did particularly well in the first stage were invited to participate in the second stage, which took place about eight weeks later in a number of venues and was monitored by the contest organizers. In our paper, we use the data from the second stage (3rd and 4th grades) and analyse the problems in which girls' performance was better than this of boys.

Gender Issue in Israeli National and International tests

In National Israeli Math tests, for grade 5 gaps were found in favor of boys (about a quarter of standard deviation on average), and it seems to be expanding somewhat over the years 2012-2017.

At the same time in National Israeli Math tests, for grade 8, the achievements of boys and girls are similar throughout the years 2012-2017. The same picture is when comparing the achievements of Israeli boys and girls in TIMSS tests (2007, 2011, 2015).

The gap in favor to boys at an average of 16 points (about 1/6 of standard deviation on average) found again in the PISA tests in mathematics literacy in years 2006, 2009, 2012 (Rapp, 2014).

Mathematical Competitions: opportunities for learning and fun

Mathematical competitions in their current form boast more than 100 years of history and tradition are organized in different formats, in different venues and for different types of students. They are considered to be "one of the main tools to foster Mathematical Creativity in the school system" (Silva, 2014). Kahane (1999) claimed that large popular competitions could reveal hidden aptitudes and talents and stimulate large numbers of children and young adults. Bicknell (2008) found the use of competitions in mathematics programs to have numerous advantages, such as student satisfaction, enhancement of students' self-directed learning skills, sense of autonomy, and cooperative team skills. Robertson (2007) reported that success in mathematics competitions, and mathematics achievements in general, seem to be linked to the love and interest instilled in students' learning experience. It also provides an opportunity to acquire high-level skills with extra training and the development of a particular culture that encourages hard work, learning, and achievement. The interplay between cognitive, metacognitive, affective, and social factors merits particular attention by researchers because it may give us more insight into the development of mathematical potential in young learners (Applebaum et al., 2013).

Among the variety of competitions, the Kangaroo Contest stands out in its main objective: popularization of mathematics with the special purpose of showing young participants that mathematics can be interesting, beneficial and even fun.

Kangaroo Contest's target population is not just the most mathematically talented students. Instead, it aims to attract as many students as possible, with the purpose of showing them that mathematics can be interesting, beneficial and even fun (see also Mellroth, 2014 for the Kangaroo contest in the Swedish context). Although, sadly, it has generally become accepted that the vast majority of people find mathematics difficult, very abstract and unapproachable, the number of contestants in the Contest proves that this need not be the case. With a huge number of competitors, the Contest helps eradicate such prejudice towards mathematics.

Choosing appropriately challenging tasks is an important condition in the successful contribution of mathematical competitions to developing students' learning potential (Bicknell, 2008). In contrast to other more challenging competitions, the Kangaroo Contest' problems are more appropriate, according to the challenging task concept suggested by Leikin (2009). Such tasks should be neither too easy nor too difficult, so to motivate students and develop their mathematical curiosity and interest in the subject.

Regarding the tasks and learning opportunities, Brinkmann (2009) mentioned that, when asked about the most beautiful mathematical problems, Grade 7 and 8 students named puzzles, while commenting that the problems should not be too difficult. For example, more than half of the students cited as 'a beautiful math problem' one of the 2003 Kangaroo Contest problems, which targeted spatial abilities in the context of paper folding (Brinkmann, 2009). Moreover, Applebaum's recent study confirmed earlier research that spatial thinking and mathematics are interrelated, especially in early grades, thus indicating that early intervention is crucial for closing achievement gaps in mathematics (Applebaum, 2017).

Kangaroo Contest in Israel

Every student who wanted to participate in the contest (possibly sometimes due to the encouragement of the students' parents) could do it without any early condition (such as a test or an interview). The students only needed to pay a very low registration fee. The students came from different parts of Israel, from large cities as well as smaller cities and villages, and from different socio-economic backgrounds.

Usually the Kangaroo contest has one test, both in Israel and in other countries. In 2016 the Israeli contest was unique, however, and involved two stages. In the first stage students coped with a test published in the internet (students needed to identify themselves) from their homes. Those who did particularly well on the first stage, were invited to participate in the second stage, which took place about eight weeks after the first stage in a number of sites and was monitored by the contest organizers.

The Study

Research Questions

In this study, we use data from the competition's second stage to investigate the following research questions:

(1) In what kind of problems girls' performance was better than that of boys?

(2) What is the difference in the choice of distractors between boys and girls?

Participants

The 158 participants (112 boys and 46 girls) aged 9-10 who took part in the final stage comprised 70 students in Grade 3, 88 in Grade 4. It should be mentioned that in Israel, there was also an online stage, which we do not take into account in this paper. However, we noticed that in the first (online) stage, the ratio of boys to girls was approximately 3:2. We cannot definitively conclude that boys were more successful than girls in the first contest stage, because the first stage was not monitored, as students solved the tasks online from their homes.

Tasks

The test lasted 75 minutes. All participants had the same version of the test that consisted of 24 problems. Using any accessories other than pens and paper was forbidden. Almost all the tasks in the Kangaroo Contest differed, in both style and type, from the tasks students usually encounter in their classroom textbooks. All tasks were multiple-choice and were ordered according to increasing difficulty (Easy – Average – High). The problems of the international contest are selected each year from a long list of problems provided by the team leaders from all the participating countries.

For each problem, a choice of five possible answers was provided. Problems 1-8 give 3 points for a correct answer and deduct 0.75 points for an incorrect answer. Problems 9-16 give 4 points for a correct answer and deduct one point for an incorrect answer. Problems 17-24 give 5 points for a correct answer and deduct 1.25 points for an incorrect answer. The point reduction for incorrect answers ensures that a completely random guess (with a probability of 20% to be correct and 80% to be incorrect) has an expectation of zero points. Not providing an answer at all gives zero points across all difficulty levels.

The results

Task	N3	N7	N8	N13	N18	N21	The Whole Test
Skills needed	Common sense	Spatial Ability	Trial and Error	Spatial Ability	Spatial Ability	Common sense	
	Maximum score: 3 points			Maximum score: 4 points	Maximum score: 5 points		Maximum score: 124 points
	Mean (St.Dev.)	Mean (St.Dev.)	Mean (St.Dev.)	Mean (St.Dev.)	Mean (St.Dev.)	Mean (St.Dev.)	Mean (St.Dev.)
Girls (N=46)	2.772 (0.878)	2.364 (1.405)	1.484 (1.834)	3.022 (2.005)	3.424 (2.693)	0.571 (2.697)	76.587 (20.566)
Boys (N=112)	2.531 (0.667)	2.344 (1.857)	1.125 (1.184)	2.536 (1.326)	2.857 (2.855)	0.558 (2.566)	81.103 (19.148)

Table 1. Tasks in Kangaroo Contest where girls had better performance than boys

Regarding the first research question, in Table 1 we present the descriptive data according to boys' and girls' performance in the final (monitored) stage of the 2016 Kangaroo Contest. According to the data presented below, we found six tasks (of 24) whereas girls performance was better that that of boys. These six tasks were evenly distributed between each one of the levels of difficulty in the test: three problems at easy level (N4, N7 and N8), one problem at average level (N13) and two problems at high level (N18 and N21). The previous analysis (Applebaum, 2017) demonstrated that distribution of tasks according to the required skills was as following: spatial abilities (tasks 7,13,18), common sense (tasks 3 and 21) and trial and error method (task 8).

Below we present the tasks where girls' performance was better than that of boys'.

Task 3

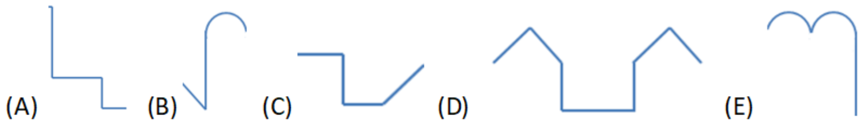
Sophie draws kangaroos: a blue one, then a green, then a red, then a black, then a yellow, a blue, a green, a red, a black, and so on... What color is the 17th kangaroo?

Blue (B) Green (C) Red (D) Black (E) Yellow

Task 7

Far away, we see the skyline of the castle:

Which of the pieces cannot belong to the skyline?

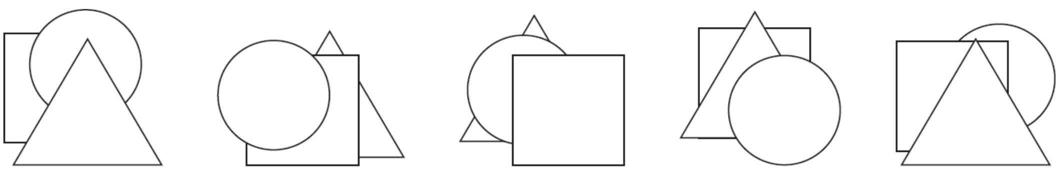


Task 8

Misha wants to circle each of the figures A - E with a felt-tip pen, not taking the pen from a piece of paper and not drawing along the same line twice. What shape could he not circle?

Task 13

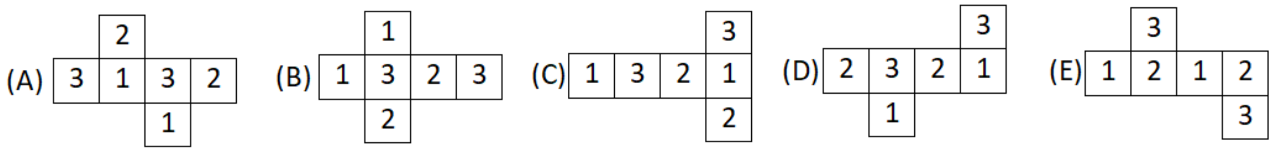
Five children had a paper square, a paper triangle and a paper circle. Every child placed their own papers in a pile, as shown in the pictures. How many children placed the triangle above the square?



(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Task 18

On each side of the paper cube, the numbers 1, 2 or 3 ate written, and the numbers on the opposite faces are the same. Which of the figures can we get if cut this cube along some edges and deploy it?



Task 21

David and Eyal count the trees that grow around the lake. They both go in the same direction but started counting from different trees. The 20th tree that counted by Eyal is the 4th tree that counted by David and the 10th tree that counted by Eyal is the 46th tree that counted by David.

How many trees are growing around the lake?

(A) 50 (B) 52 (C) 56 (D) 60 (E) 80

Regarding the second research question, in Table 2 we present the descriptive data according to boys' and girls' distractors' choices in above mentioned tasks (the correct answers were shadowed).

Task		A	B	C	D	E	No answer
N3	Girls	0	4.35	2.17	91.3	0	2.17
	Boys	1.79	0.89	3.57	88.39	5.36	0
N7	Girls	4.35	8.7	82.61	0	2.17	2.17
	Boys	5.36	8.04	82.14	1.79	0.89	1.79
N8	Girls	8.70	6.52	56.52	10.87	13.04	4.35
	Boys	4.46	4.46	50	12.5	24.11	4.46
N13	Girls	0	0	19.57	80.43	0	0
	Boys	1.79	0.89	25.89	70.54	0	0.89
N18	Girls	2.17	10.87	6.52	2.17	73.91	4.35
	Boys	3.57	6.25	8.04	7.14	63.39	11.61
N21	Girls	2.17	26.09	28.26	15.22	13.04	15.22
	Boys	6.25	24.11	23.21	6.25	16.07	24.11

Table 2. Boys' and girls' distractors' choices (in percent)

No significant gender differences were found in the selection of distractors, except for the case of Task 8 in which more than 24 percent of boys chose the distractor D, compared to 13 percent of girls. This point can be explained by the fact that distractor D was the most complex and boys bet on choosing it.

Short Discussion

The gender issue in mathematics, i.e., girls being underrepresented in the STEM-related fields, still remains unresolved. This is why every inclusive endeavor to popularize mathematics by attracting all students merits particular attention. Kangaroo contests are exemplary of such inclusive competitions. With limited research available on the patterns of participation and the results of the contest, it is important to investigate gender-related issues. We analyzed the results of participants from Grades 3-4 in the 2016 Israeli Kangaroo contest, according to gender, and found that boys generally performed better than their female counterparts. However, we realized that gender differences are task-dependent: in our study, 6 tasks (of 24) were solved more successfully by girls. Among the tasks on which girls overperformed boys, solutions in 3 tasks targeted spatial abilities, in 2 tasks were directed at common sense and in 1 task – trial and error method. Contrary to studies (Halpern, 2011; Kimura, 2000) that indicated significant differences (in favor to boys) on spatial abilities, we found that the performance of girls was better than that of boys when coped with 3 tasks target spatial abilities. The data do not yield any far-reaching conclusions about the factors that might explain these findings. Yet, it is worthwhile to conduct further research and analysis over the next few years, other grades and to see if the pattern re-appears. Furthermore, deeper analysis is needed regarding the tasks that were solved better by girls and the methods they used in solving them. We plan further examination on students' basic cognitive traits and thinking styles in association with their success in solving different types of tasks in Kangaroo contest. There is a possibility that gender differences are linked to these personal characteristics (Sladek, Bond & Phillips, 2010).

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WHAT DO STUDENT TEACHERS BELIEF ABOUT MATHEMATICAL GIFTEDNESS? FIRST INSIGHTS OF AN EXPLORATORY STUDY

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Abstract. *What do teachers or student teachers belief about certain pedagogical and didactic, subject or curricula related facts? Or what might characterize their individual knowledge? Both beliefs and knowledge guide teachers as to their acting in classrooms, for example, regarding the support of different groups of children such as mathematically gifted. This is why professional knowledge bridging objective and individual constructed facets is considered a fundamental base for both teachers' and student teachers' training and education. The article presents first insights of an exploratory study focusing on German student teachers' beliefs about mathematical giftedness.*

Key words: Professional knowledge; beliefs; mathematical giftedness.

INTRODUCTION AND RATIONALE

Studies on the knowledge of teachers or student teachers are a comprehensively considered desideratum, especially as to creating a basis for their education and training. In the relevant literature, components of professional knowledge (such as pedagogical content knowledge) are deemed comparatively well identifiable. In addition, 'beliefs' which can be described as the convinctions of an individual as to a certain object, but which from an objective perspective do not necessarily have to correspond, for example, to the state of research and rather represent an alternative subjective construction are felt to be just as guiding for (e.g., pedagogical) action as knowledge itself (for a survey: Schwitzgebel, 2015). In contrast to knowledge, beliefs and comparable concepts are an unsharp construct, and beliefs' operationalization is much more difficult. Because objectively shaped knowledge and subjectively shaped beliefs are also considered indistinguishable from each other (in summary: Pajares, 1992), some current approaches combine beliefs with knowledge in a narrower sense in the framework of professional knowledge (DZLM, 2015; Kuntze, 2012). In current mathematics education research as well as in related disciplines, there are numerous studies on both constructs and their synthesis with different focuses. For example, according to Philipp (2007), as to mathematics teachers' beliefs, four main research areas can be characterized, namely research about the mathematical thinking of pupils, about mathematical curricula, about technology, and about gender. Regarding pupils' mathematical thinking, teachers' or student teachers' beliefs as to mathematical giftedness provide a research focus which had not been studied very comprehensively yet; nevertheless, it determines an important basis with regard to their education and training, and as a consequence with regard to their acting in identifying and supporting mathematically gifted pupils, which also seems to apply to the holistic connections to professional knowledge as outlined above. In this article first insights into a study are given, that focuses on this desideratum by investigating the question how frequent characteristics of student teachers' beliefs about mathematical giftedness can be described. First, brief overviews of the theoretical frameworks of both beliefs (under the umbrella of professional knowledge) and mathematical giftedness will be given. Second, the study's design and its first results will be subsumed and discussed.

THEORETICAL BACKGROUNDS – BRIEF OVERVIEWS

Beliefs and their relationship to knowledge

'Beliefs' are named as a 'messy construct' (Pajares, 1992). In the recent past, for example, 'beliefs' have become a kind of container concept in approaches of mathematics education's research, which is able to conceptually sketch the origin and functions (e.g., Rolka, 2006), but which complicates the determination of specific components and makes overlaps unavoidable – in particular, to knowledge as a different construct.

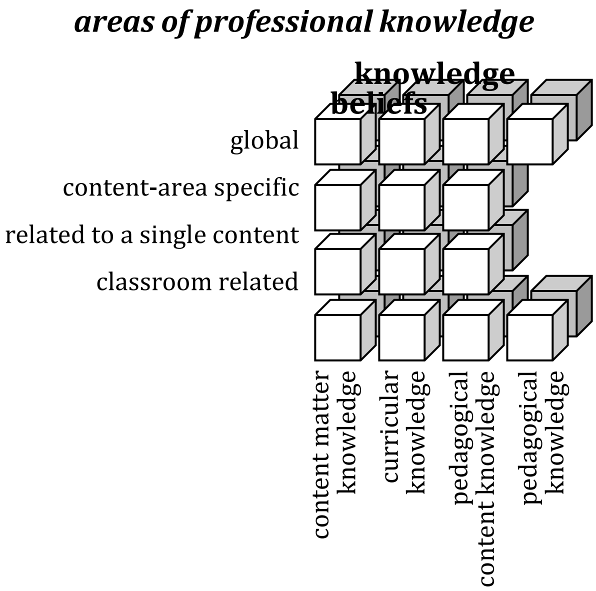

Figure 1: Theoretical framework connecting both beliefs and knowledge (following Kuntze, 2012)

As already indicated, the term 'professional knowledge' touches the interface to 'beliefs' (or 'convictions' and similar constructs), and a precise delimitation is considered extremely difficult to determine (in summary: Pajares, 1992). Irrespective of whether one focuses more on 'knowledge' or 'beliefs' (or constructs such as convinctions or attitudes that can hardly be distinguished conceptually), there is a very large number number of publications. Starting from professional knowledge, beyond pedagogical knowledge, an established approach distinguishes between 'subject matter knowledge', 'pedogogical content knowledge' and 'curricular knowledge' as three necessary dimensions (Shulman, 1986). In this sense, pedogogical content knowledge refers to the knowledge of possibilities to teach subject related contents. According to Ball, Thames and Phelps (2008), such knowledge also includes, for instance, knowledge about student perspectives on specific topics. Independent of certain operationalizations there is a scientific consensus that pedogogical content knowledge forms a determining moment in the professional knowledge of teachers and a bridge between their subject matter knowledge and their teaching activities (Brown & Borko, 1992). In the relevant research, as a result of the delimitation problem, cognitive aspects (such as objective knowledge that can be assessed by experts as true or false) and subjectively developed aspects (such as beliefs that do not have to agree with knowledge that is objectively assessed as true or false, i.e. in the sense of a complement) are sometimes triangulated in model theory, often also in conjunction with affective components such as self-efficacy expectations (e.g., DZLM, 2015). Consequently, pedogogical content beliefs are interpreted as an equivalent to pedogogical content knowledge (Kuntze, 2012). Thus, such beliefs focus on handling specific didactic situations like identifying and supporting mathematical giftedness. Both cognitive and subjectively developed aspects influence an individual's behavior as to, for example, diagnostics and support (Fischer, Rott, & Veber, 2015). The framework model of the present study follows the idea of combining both dimensions (cf. Figure 1), because it offers a holistic basis for the intended reconstructions of student teachers' beliefs on mathematical giftedness by interpreting both cognitive and in particular subjective facets, including aspects of self-efficacy, whereby 'professional knowledge' provides an umbrella term.

In addition to the introductory outlines of the study's relevance, according to Sowder (2007), six main objectives can be identified in order to determine important cornerstones of conceptions as to teachers' professionalization, which allow a deeper classification with regard to research on beliefs about mathematical giftedness:

- Clarifying which knowledge and which skills should be included in the professionalization of teachers, in particular
- which mathematical-content knowledge is important and to what extent.
- the development of abilities regarding the analysis of children's thinking and learning as well as
- in general, the development of pedagogical content knowledge.
- the development of an understanding of equality as a principle, i.e. the development of a view of all pupils in the design of teaching-learning processes.
- the development and reflection of one's own identity as a mathematics teacher.

Research on these main objectives and in particular the elaboration of cornerstones as to necessary professional knowledge continues to be a main focus of mathematics education research. Among other things, this includes professionalization in the context of identifying and supporting mathematically gifted children, which can be classified in accordance with both the first and the fifth of the points listed above.

Mathematical giftedness

Recent approaches that describe giftedness indicate a scientific consensus on the following aspects (e.g., iPEGE, 2009; cf. Benölken, 2015): First, it is a complex phenomenon, and, thus, it is necessary to consider both cognitive and co-cognitive intra- and interpersonal aspects. Second, giftedness is felt to be a domain-specific phenomenon, and, for example, domain-specific criteria of mathematical giftedness have been found (e.g., Käpnick, 1998; Sheffield, 2003; Assmus, 2018). Third, concurrent models distinguish an individual's potential clearly from his or her performance, and, therefore, mathematically gifted children should be identified and fostered as early as possible. Consequently, giftedness can be seen as a dynamic phenomenon calling for a holistic view on an individual's personality and, therefore, long-term process-diagnostics. The aspects mentioned above are synthesized within recent modeling of mathematical giftedness (Nolte, 2012; Fritzlar, 2015), which for the most part follow general models of educational sciences (e.g., Gagné, 2000) from a specific perspective of mathematics education. The study's framework to interpret student teachers' beliefs as to mathematical giftedness considers the aspects of concurrent modeling of mathematical giftedness, understanding 'mathematical giftedness' as an above-average potential as to specific criteria, characterized by individual determinants and a dynamic development depending on inter- and intrapersonal influences (cf. Figure 2).

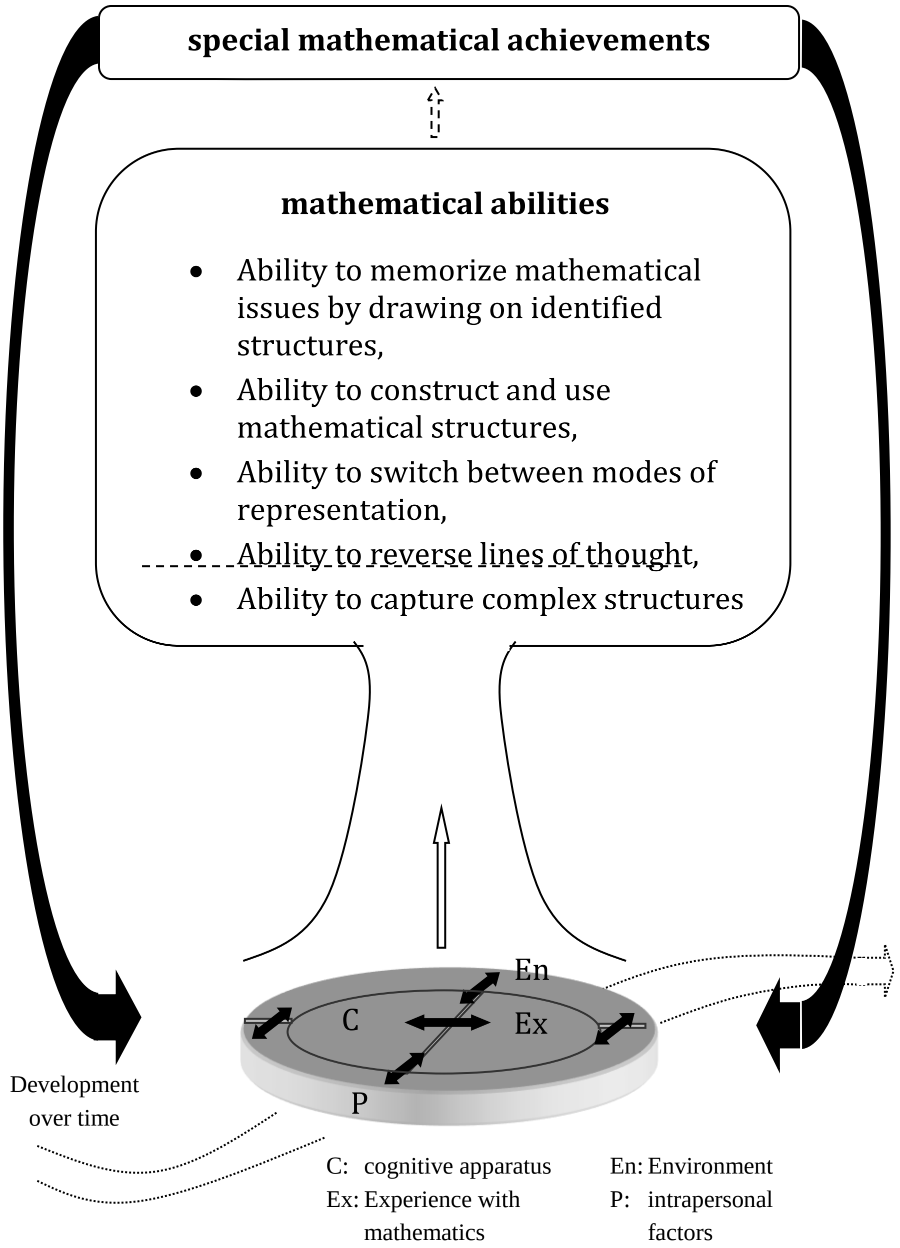

Figure 2: Model for the development of mathematical giftedness (following Fritzlar, 2015)

THE STUDY

The study focuses on the question how frequent characteristics of student teachers' beliefs about mathematical giftedness can be described. The participants were N=306 (275 f., 31 m.) primary student teachers at the University of Halle-Wittenberg and the University of Wuppertal. At the time of questioning, nearly 200 of the student teachers were in the third year of their academic studies, the others were in the first or second year. The study's character is explorative, i.e., generalizations were not intended, but existential propositions (Lamnek, 2010). Thus, a qualitative design was advisable. The data have originated from questioning at the very beginning of the summer semesters 2017 and 2018. As to the method, qualitative data were generated by open questionnaires, which were anonymized to ensure unbiased interpretations. The questionnaires were fulfilled in written form. Following demands of the construction of questionnaires (like introductory remarks on the subject of questioning), the student teachers were asked *“What do you think constitutes mathematical giftedness?”* (translated from German), and some ideas in the form of questions where added below (for example, *“Which characteristics are typical for mathematical giftedness?”*, or *“How do you recognize mathematical giftedness?”*). The questionnaire and all procedures where tested within a small pilot study with two probands before. As to the analysis, in accordance with the approach of inductive category creation within a qualitative content analysis (Mayring, 2010), emerging from the authentic data, codes were developed using the software “MAXQDA” to find characteristic categories of student teachers’ beliefs about mathematical giftedness (also as basis for a possible later typing). The following characteristic procedural steps were observed: (1) initiating text work, (2) developing main categories, (3) coding the data with the main categories, and (4) determining subcategories (Kuckartz, 2016).

In a very first phase, which is reported in this article, a subsample of 21 participants was considered. They attended a seminar at the end of their academic studies. In some cases, they had already met the topic of giftedness in the context of seminars or in practical phases. Thus, it could be assumed that their responses reflect a wide range of different perspectives. As a consequence, the resulting category system seems to provide a suitable basis for the second phase of analysis, i.e. the application and differentiation of the categories in the overall sample. Against this background, the authors progressed on their own to built categories in a first step, and they compared their interpretations in a second step. Afterwards, categories which had been found were applied to the subsample again in order to verify their adequacy, and subcategories were determined. The further procedural steps (5) application and differentiation of the category system in the overall sample, and (6) complex analyses will be part of a second phase of evaluation.

RESULTS – FIRST INSIGHTS

The following report only considers evaluation results from the first phase, since the analysis is currently been conducted in the overall sample.

Table 1 shows both categories and respective subcategories which inductively have been formed and systematized based on the data under consideration (the numbers in square brackets are included in the analysis of a case example presented below). Of course, formal definitions were provided: Table 2 presents an example of the subcategories’ definitions as to the category “cognitive indicators”. Both the phrasings and the quotes shown in the Tables 1 and 2 were translated from German.

category	Subcategories	
basic assumptions for the definition	origin of giftedness: genetic disposition	
	origin of giftedness: genetic disposition and influence of interpersonal catalysts	
	origin of giftedness: genetic disposition or influence of interpersonal catalysts	
	origin of giftedness: genetic disposition and influence of interpersonal as well as intrapersonal catalysts [5]	
	above average in comparison to others (as to, e.g., achievement) [1]	
	domain-specific phenomenon [2]	
cognitive indicators	criteria of mathematical giftedness in a narrower sense	
	indicators focusing on mathematical general education	
	general indicators [3]	
co-cognitive indicators	enthusiasm for mathematics	
	perseverance	
	high concentration	
	self-reliance	
psychosocial consequences	affective or emotional consequences (e.g., boredom, underchallenge) [4]	
	behavior-related consequences (e.g., disrupting lessons, frequent requests)	
identifying giftedness	individual approach	
	social perspective: Comparison with others	
	standardized tests	
	diagnostic competence necessary basic prerequisite	
supporting and fostering mathematical giftedness	qualitative [8]	inside school
	quantitative [9]	outside school
	socially [6]	accelerated
	in class [7]	motivational

Table 1: Categories und subcategories reconstructed within the first interpretational step

subcategories	Definition	examples
criteria of mathematical giftedness in a narrower sense	Includes aspects that consider characteristics of mathematical giftedness regarded as established knowledge in the literature (e.g., recognition of structures).	*“it is easier for them to find mathematical structures and patterns”*
indicators focusing on mathematical general education	Includes aspects assigned to mathematical general education (e.g. contents of curricula).	*"there is an understand-ding of numbers, but also forms, etc."*
general indicators	Includes aspects concerning general cognitions (e.g., quick comprehension).	*“quick comprehension of tasks”*

Table 2: Definition of subcategories for the category “cognitive indicators”

As an example, the assignment of the categories will be clarified by the text of a 21-year-old student teacher (the numbers in square brackets correspond to the assignments of Table 1). The person wrote (translated from German; of course, the original data can be requested from us): *“Mathematical giftedness is a pupil's above-average ability [1] in the field of mathematics [2]. This means that pupils find very complex solutions to open tasks [3] and they are underchallenged in frontal teaching [4]. The underchallenge can be seen, for example, in inattentiveness or preoccupation with other things. Mathematical giftedness could arise from a high interest in mathematics at an early age or from parents who teach their children important contents before they go to school. Furthermore, such kind of giftedness could also be genetic/hereditary [5]. Such children, who are mathematically gifted, can, for example, play an explanatory role in class and help weaker pupils to cope with tasks [6], [7]. Furthermore, one can work a lot with puzzle tasks [8] or additional tasks [9] in the classroom in order to do satisfy the children's heterogeneity.”*

It is remarkable that the student teacher mainly focuses on the characterization of general indicators (very complex solutions) and affective consequences (underchallenge). In contrast, the person does not mention criteria of mathematical giftedness in a narrower sense as described in literature (e.g., the recognition and use of mathematical structures) or indicators focusing on mathematical general education (e.g., calculation skills). Also in the other questionnaires criteria of mathematical giftedness are rarely mentioned, but indicators focusing on mathematical general education are often taken into account.

DISCUSSION AND PERSPECITVES

Due to the small sample and the early stage of analyses, generalizing statements currently are not possible, even if the material analyzed so far suggests interesting impressions, as they can also be found in the reported questionnaire example: student teachers, for example, seem to be more likely to attach mathematical giftedness to facets of mathematical general education, i.e. their individual beliefs do not meet models of mathematical giftedness representing objective knowledge. Thus, implementing specific contents in the mandatory curricula of their academic studies would result as a necessary practical consequence to ensure an adequate support of mathematically gifted pupils in their later acting as teachers. Of course, this is nothing more but a possible impression of the first study’s phase. Moreover, the design of the entire study is explorative and it has obvious limitations; e.g., the reconstructions were conducted by two researchers, but it remains uncertain, if a consensus view is the right one (Lamnek, 2010). Another limitation might result from the fact that the data collection was conducted in written form, since it is possible that the necessary amount of time may make more detailed processing more difficult. Therefore, a long-term goal is a study with both individual and group interviews that could provide more differentiated impressions.

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DISCERNING TWO CREATIVE ACTS: EXPANDING POSSIBILITIES AND DIVERGENT THINKING

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Abstract. *This study is part of a broader research study exploring collective creative acts in elementary mathematics learning environments. In this paper we use two metaphors—expanding possibilities and divergent thinking—to describe two types of learning acts of a group of sixth grade students while they are working on a mathematical task. We then use examples from the group's acts to discern some similarities/differences between the two metaphors. We acknowledge that students' acts under both metaphors are important and desired learning acts that should be promoted and sustained in mathematics learning environments. However, teachers' awareness of such discernments would promote their abilities and professionality in being alert and responsive in their interventions within the collective to further the evolving structure of mathematical thinking and understanding.*

Key words: Creativity, Collective Creativity, Expanding Possibilities, Divergent Thinking.

INTRODUCTION

Based on the findings of a broader research study exploring the nature of collective creativity in mathematics learning, Aljarrah (2018) suggested four metaphors to describe students' creative acts while they are working on assigned mathematical tasks: summing forces, expanding possibilities, divergent thinking, and assembling things in new ways. "Summing forces" can be used to describe learners' collective effort to confront a problem and to decide where to start and how to proceed; "expanding possibilities" might be understood as broadening learners' horizon by gaining new insights based on their previous insights; "divergent thinking" requires students to diverge outside their known content-universe, outside their safe zone of acting and thinking, outside the problem's clearly given conditions and information, and even outside the content of the planned curriculum; and "assembling things in new ways" can be considered as a metaphor of creativity that implies looking for associations and making connections.

During all phases of the broader research study, the question of how to differentiate between expanding possibilities and divergent thinking actions attracted a lot of debate. In responding to such questioning, we used examples of learners' actions to discern some similarities/differences between the two metaphors and in this paper we present these examples and discuss how teachers might shape teaching practices to foster these kinds of thinking.

LITERATURE REVIEWC

"Divergent thinking" is one of the most popular expressions used in the literature to describe creativity, as reflected in Webster's online dictionary that characterizes creative thinking in terms of following many lines of thought in order to generate new, original solutions to problems. In the field of mathematics education, one can easily recognize that the majority of research in mathematical creativity frames creativity in a manner consistent with this term or its synonyms (e.g., Haylock, 1997; Levenson, 2011; Silver, 1997).

It is possible that, because mathematical creativity is associated with problem solving abilities such as fluency, flexibility, elaboration, and originality, that it is seen as a synonym for divergent thinking. Researchers who are interested in mathematical creativity usually use these abilities as components of mathematical creativity (e.g., Kattou, Kontoyianni, Pitta-Pantazi, & Christou, 2011). Haylock (1984) claimed that "various kinds of divergent productions tasks can be devised in mathematics that generate responses that can be judged by such criteria as flexibility, originality and appropriateness" (pp. 68–69). On the other hand, Craft (2001) claimed that creativity (little-c creativity) in learning environments enables learners to generate and expand ideas, to suggest hypotheses, to apply imagination, and to look for alternative, not-yet imagined approaches. Craft (2000) argued that one of the engines for little-c creativity is the idea of "possibility." She conceived of the idea of "possibility" as using imagination, asking questions, and playing. And, according to Davis (1996), "the function of playing is to open a space of possibilities….Creativity arises out of the background of those possibilities, selecting out through repetition and formulation those actions that are new and useful" (p. 220). These ways of describing creativity in mathematics learning environments seem more consistent with the "expanding possibilities" metaphor.

METHODOLOGY AND METHODS

This report is part of a broader research study, led by the first author, exploring collective creativity in elementary mathematics learning environments. To fulfil the objectives of the study, we adopted a design-based research methodology (DBR) (Design-Based Research Collective, 2003).

Two mathematics teachers, and 25 of their sixth-grade students in a Canadian school setting participated in the study, in collaborative problem-solving sessions inside and outside their classrooms. The video-recordings of these group activities formed the core of the data. Following Powell, Francisco, and Maher's (2003) analytical model, the first step of analysis was to watch every video in order to familiarize ourselves with its content. The next step was to choose the most significant videos and to prepare written, time-coded descriptions of their contents. In the analysis and interpretation of the data, the focus was on the group's (co)acting and interacting at moments of creative mathematical expression.

DISCERNING STUDENTS' CREATIVE ACTS

Here, we use two metaphors—expanding possibilities and divergent thinking—to describe two types of learning acts of a group of sixth grade students while they are working on a mathematical task. We then use examples of the group's actions to discern some similarities/differences between the two metaphors. The examples are taken from a 25-minute problem solving session/interview that features a group of three students, Zaid, Mark, & Kyle (pseudonyms) with the first author. The session took place in a quiet room in the Student Centre in the participating school. The first author introduced a task to the group (adapted from an example in Empson and Levi, 2011) and asked them to work on it together:

Three children, Alex, Zac, and John, shared a chocolate bar. Explain in as many ways as you can how those children may divide the chocolate bar into three pieces such that Alex will get twice what John got, and John's part is no more than one-fourth of the original bar and no less than one-tenth of it.

The students had been described by their teacher as high-ability students who usually work individually in class. We use a combination of both our voices and the three students' voices to describe their doings and communications which, for us, are the essence of their creativity. (Note: In our transcript we use dashes to show an interruption of one speaker by another).

Creativity as expanding possibilities

Here, expanding might be understood as moments when the group broadens its mathematical horizon by gaining new insights based on the group's previous insights. It is a kind of stretching the space of the possible as a result of the evolution and growth of the group's insights. In this case, any new insight is dependent and contingent upon the group's previous insights. Here, we present just one example of students' practice of expanding possibilities.

After the group (i.e., Zaid, Mark, & Kyle) engaged in an effective, interactional conversation, they, deliberately, started to relax the 1/4, 1/10 constraint in the problem, which allowed them the space they needed to realize the potential of infinite chopping in a less-encumbered space. Kyle summarized their different basic options: *"Okay, so, so we have our ninths, and we have our eighths, now sevenths, sixths, and fifths, yeah, these are our options for that."* Zaid stressed the idea of *going on forever*: *"You see, it goes on forever."* Mark agreed with him: *"Yes it does. We could do, um, we could go on forever."* At that point, the task the students set for themselves shifted from finding as many ways as they could to divide the chocolate bar into three pieces under specific constraints to proving that the processes of dividing the chocolate bar under such constraints would go on forever.

The problem-solving session with this group of students can be described as one of "expanding combinatorics" (Sawyer, 2003, p. 7). For example, the group used their *realistic* options to divide the chocolate bar (i.e., what might be possible with the physical constraints of a real chocolate bar) to generate and explain their *mathematical* ones, which according to them *go on forever*. They considered tenths, ninths, eighths, sevenths, sixths, and fifths as their *realistic* options for dividing the chocolate bar. On their working sheet they wrote, *"last section always divided by four once you reach the last realistic idea."* Through their discussion, one can understand what they meant by this conclusion. For example, one of their options was to give John two-ninths, Alex four ninths, and Zac three-ninths. They expanded this realistic option based on imagination and playful activities (e.g., the interplay between the realistic and the mathematical, and the using of the 'zooming in' expression to imagine and to justify the possibility of going on for ever). Kyle suggested, *"As long as John always gets half of what Alex gets, Zac does not really matter."* Since Zac's portion could be anything, they could take one of his parts, divide it into four equal-sized sections, and give one to John and two to Alex and then divide *that* last section (i.e., the one taken from Zac's) into four, give one to John and two to Alex, and so on; they could keep *zooming in* to the last section and do the same thing an infinite number of times. According to them, this idea can be applied to all *realistic* options (i.e., the tenths, ninths, eighths, sevenths, sixths, and fifths).

During the whole problem-solving session with this group of students, they were eager to discuss and expand a variety of potential possibilities, and willing to take their initial ideas to new heights. But before they could expand and build on their initial ideas and thoughts, they negotiated their understanding of the conditions of the problem, thereby breaking with an initial assumption indicated by Mark's statement, "*I do not think it can go forever. We cannot go more than one-fourth and we cannot go less than one-tenth.*" We believe that the group's deliberate relaxing of the 1/4, 1/10 constraint was critical in expanding the space within which many new possibilities started to emerge and evolve. With each mutually specifying action, the space of the possible was enlarged, ultimately reaching a size that engulfed (and perhaps surpassed) that which was needed to solve the initial problem. Likely, as it grew, that space of possibility engulfed many other problems that might have been posed.

Creativity as divergent thinking

Although there is some commonality between *expanding possibilities* and *divergent thinking*, we consider expanding possibilities as building on and expanding what has already been developed and sustained (ideas, processes, concepts, conditions etc.), whereas divergent thinking requires students to diverge outside their known content-universe, outside their safe zone of acting and thinking, outside the problem's clearly given conditions and information, and even outside the content of the planned curriculum.

We believe that the group's divergent thinking evolved while they were reflecting on and verifying their realistic ideas, and where they were trying to reason about their *mathematical* ones. The students were pretty sure that there was a mathematical way to show that the process of dividing the chocolate bar can *go on forever*. Here are their voices while they were trying to sustain their strategy for the task:

Kyle: *So, we can do a list of the realistic ones by doing this [while he was pointing to some of their representations of the realistic options].*

Mark: *Yeah, let's do the realistic ones, because I am not going into 'Zac gets one molecule'.*

Zaid: *Then, if it is hard for us to keep going on forever, then just say it goes on forever, and then explain our reasons.*

Their comments, gestures, and representations reflected their divergent thinking and their emergent ability to connect and move between the mathematical (sometimes they called it the technical) and the realistic. For example, Kyle reflected on and expanded his initial suggestion: "*Okay, so, so we have our ninths, and we have our eighths, now sevenths, sixths, and fifths, yeah, these are our options for that*," and noted, *"There is a way in between them [he meant between any two of these different possibilities]."* His statement "*There is a way in between [any two realistic possibilities]*" was an incident where the group went beyond the stated conditions of the problem and started to think out of its content universe. Mark continued, *"Yeah, mathematically this will continue forever."* Kyle agreed, and Mark continued to sustain the group's suggestions by stating, *"Realistically, no, no, you cannot give someone half of—"* Kyle interrupted Mark and completed his statement by stating, *"Yeah, you have to stop at one point."* At that point, the interviewer (the first author) intervened subtly by reorienting the group's attention to try to use the number line. So, Mark initiated a new conversation about using the number line and decimals to explain why, *"mathematically"* the process of dividing the chocolate bar could *"go on forever"*:

Mark:	*You could, you could also put it in decimals.*
Zaid:	*But, I mean if you did decimal fractions then you can do anywhere from here [indicating 0.1 on the number line] to all the way to [indicating 0.25]—*
Mark:	*Look, if you did zero point two five, to zero point one—[drawing a line segment to represent a part of the number line with 0.1 and 0.25 as its endpoints]—*
Zaid:	*And, twenty-five hundredths, and umm [while he was writing on a piece of paper].*
Mark:	*Yeah.*
Zaid:	*If you used the decimal fractions there will be a loooooot.*
Mark:	*If you did this that is endless. You could do—*
Zaid:	*Oh, yeah endless, because you could just keep adding like...point one, point one, point one, point one, one, one, one, one, one, one, one, one, one...[he wrote 0.1000000000000001 on one of their shared pieces of paper]—*
Kyle:	*You could keep going zeros, um, all the way.*
Mark:	*That is technically more, um—*
Zaid:	*Or you could just keep going one, one, one, one, one, ... (0.111111111...), because if you do point one, if there is two ones it is a bigger number by, umm, one-hundredths.*
Mark:	*Exactly, for decimals it is endless.*

DISCUSSION AND CONCLUDING THOUGHTS

A key similarity between divergent thinking and expanding possibilities activities is that both require being imaginative. In fact, Vygotsky (2004) suggested that all forms of creativity are based on imagination, and according to him this is what distinguishes creativity from other forms of human activities. In discerning between divergent thinking and expanding possibilities, we turn to the grounding metaphor for each. Expanding possibilities draws on a grounding metaphor of inflation while divergent thinking is grounded by the notion of multidirectionality. Hence, we see in the first example that the students seemed to be inflating their initial ideas and thoughts (they restricted themselves to the use of fractions) to show that the dividing process could go on forever, while in the second example they seemed to be looking into multiple directions (i.e., the use of number line and decimals) to find a more sophisticated (for them) explanation of their claim. To assist students in expanding possibilities, the teacher should be ready to extend possibilities by stretching the space of presented possibilities (Aljarrah, 2018). To assist students in divergent thinking, the teacher should be ready to re-orient attention to aspects of the problem that are ripe for alternative interpretations. Of course, both of these teaching actions demand that the teacher understand the relevant mathematics deeply and have the flexibility of practice to be able to see new possibilities and alternative interpretations and be able to act in the moment to cultivate them.

Most experts in mathematics education would support calls for teaching for creativity and promoting creativity in mathematics learning environments, yet teachers and mathematics educators continue to struggle with converting such calls into pedagogical initiatives that can be implemented in the classroom (Aljarrah, 2018). We believe that teachers' abilities to recognize students' creative acts, and to discern some similarities/differences between such acts would promote their abilities and professionality in being alert and responsive in making "judgments [within the collective] about when and how to intervene" (Martin & Towers, 2011, p. 274) to further the evolving structure of mathematical thinking and understanding.

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'LEMAS' – A JOINT INITIATIVE OF GERMANY'S FEDERAL GOVERNMENT AND GERMANY'S FEDERAL STATES TO FOSTER HIGH-ACHIEVING AND POTENTIALLY GIFTED PUPILS

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Abstract. *In contrast to other countries, the proportion of high-achieving pupils in international comparative studies in Germany is relatively low. Therefore, Germany's federal government and Germany's federal states have launched an initiative in which an interdisciplinary network of scientists together with schools develops guiding principles and adaptive concepts to support achievement. This article gives an initial insight into the structure, aims and projects of the initiative, in particular into a subproject on mathematics.*

Key words: support program; high-achieving pupils; capable pupils; gifted pupils.

INTRODUCTION

International comparative studies show that relatively few pupils in Germany can be classified as particularly high and relatively many pupils as poor achievers compared to other countries. In the PISA study of 2015, for instance, 17% did not reach the basic competence level, which is slightly below the OECD average of 23%. 13% were assessed to be particularly high-achieving, which is slightly above the OECD average of around 10%. Much larger proportions of particularly high-achieving pupils were found, for example, in Singapore (about 30%) or in South Korea, Switzerland and Canada, mostly in connection with a comparatively small proportion of low-achieving pupils (e.g. Hammer, Reiss, Lehner, Heine, Sälzer, & Heinze, 2016). The top 5% of pupils at German primary schools achieved significantly lower in mathematics than pupils in comparable industrial nations such as the USA (Selter, Walter, Walther, & Wendt, 2016). While the outcomes of pupils in the lower range of achievements in Germany have continuously improved over the years, the proportion of the top groups in both primary and secondary schools has remained almost unchanged, indicating that the potential of many pupils is not recognized or adequately supported (Uhlig, 2010). Nevertheless, particularly high-achieving or capable (in the sense of capable for high achievements) pupils often have very different characteristics of, for instance, giftedness and learning needs. For example, there are pupils with accelerated cognitive, but age-appropriate social development (e.g., Roedell, Jackson, & Robinson, 1989), and gifted girls with often disadvantageous motivational determinants such as mathematical self-concepts (Benölken, 2014). Such phenomena lead to immense challenges for schools and teaching, because the claim of educational policy guarantees equal opportunities for all pupils, including those who are particularly high-achieving or capable, regardless of their origin, gender, or social status.

In Germany, educational policy is primarily the responsibility of the federal states, which have established different cultures as to, for instance, the support of particularly high-achieving or capable pupils. All the more remarkable is the joint initiative 'LemaS' ('Leistung macht Schule', an allegory combining different meanings in the German language like 'achievement produces schools as well as it becomes presentable') of Germany's federal government and Germany's federal states, funded by the Federal Ministry of Education and Research from 2018 onwards for a total of ten years.

The initiative has set itself the aim of improving the support of particularly high-achieving or capable pupils across all school subjects. Furthermore, on this basis, according to the allegory “A rising tide lifts all ships” (Renzulli, 1998, p. 105), all pupils independent of specific facets of diversity should be supported, and the general question is: How can all pupils be supported by taking the perspective of supporting high-achieving and capable pupils? Thus, LemaS represents an extraordinary school development project for the national context that has the potential to have a lasting impact on the educational landscape. Very remarkable are the extensive organization connecting some hundred schools and an interdisciplinary research group, the cooperation between the federal government and the federal states (which is very rare in questions of education), and the approach from the perspective of fostering achievement and capability, since the traditional German school system tends more towards deficit orientation and homogenization.

This article presents first impressions of the interdisciplinary theoretical foundation of the initiative as well as of both its structure and aims. In addition, a first insight into a subproject on mathematics is given.

OUTLINE OF THE THEORETICAL BACKGROUND AND APPROACHES

Figure 1 outlines the theoretical framework of LemaS that is a result of discussions between scientists from different disciplines who apply different and sometimes divergent terms in the context of describing phenomena related to achievement (both more detailed sources and explanations can be retrieved from https://www.lemas-forschung.de).

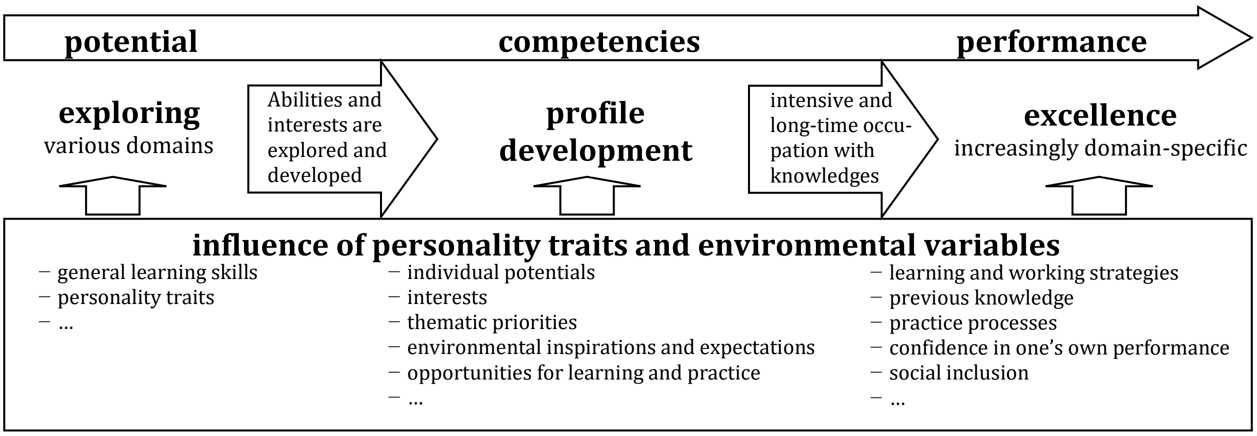

Figure 1: LemaS-framework on the development of achievement.

The umbrella term *achievement* is understood as multidimensional and developmental, which also is indicated by the association of capability for achievements. On the one hand, the construct is regarded as school-related achievement, but on the other it also includes personality development and social responsibility. It is assumed that achievements can be produced in all domains, which are of importance in society, such as natural sciences, languages, music, art, mathematics, or social-emotional and ethical-philosophical areas. A domain-specific distinction is made between abilities (‘can do’) and personality traits (‘will do’). Under the umbrella term achievement, the term *potential* is used to describe general individual dispositions, and the term *competencies* is applied to describe domain-specific dispositions.

It is assumed that potentials and competencies develop dynamically, influenced by interpersonal and intrapersonal factors. In contrast, the term *performance* is applied to describe observable abilities or skills in the different achievement domains, and the term *excellence* represents high performance. In addition to these fundamental distinctions, some subprojects do not focus on achievement or capability, but on the specific construct of *giftedness*. Against the background of the complex formed by the terms mentioned under the umbrella of achievement (cf. Figure 1), 'giftedness' and 'potential for excellence' are equated to declare the relation between those concepts. Thus, the holistic domain-specific perspective on achievements' development allows an interpretation of the term giftedness in accordance with current modeling of giftedness, in particular mathematical giftedness (e.g., Fuchs & Käpnick, 2009). Giftedness is a prerequisite of excellence, but not identical to it (DLR, 2018a).

According to this definition, support is necessary for the development of achievement. Appropriate domain-specific support concepts have to include both competencies and personality traits. This understanding has important consequences for the development of adaptive support concepts: When developing such concepts, it has to be considered that pupil's potentials vary, and that the development of achievement is an individual process in which supporting inter- and intrapersonal influences can differ between individuals. Thus, concepts of support have to be organized in such a manner that individuality is taken into account. Moreover, while offers have to be created in the beginning of a support process, competencies or personality traits in motivational, volitional, emotional, and social interdependencies become increasingly important in the process. Therefore, schools should offer spaces for the development of potentials and achievements, i.e. for comprehensive educational processes of pupils. The development of both suitable guiding principles and concepts is the starting point of the initiative.

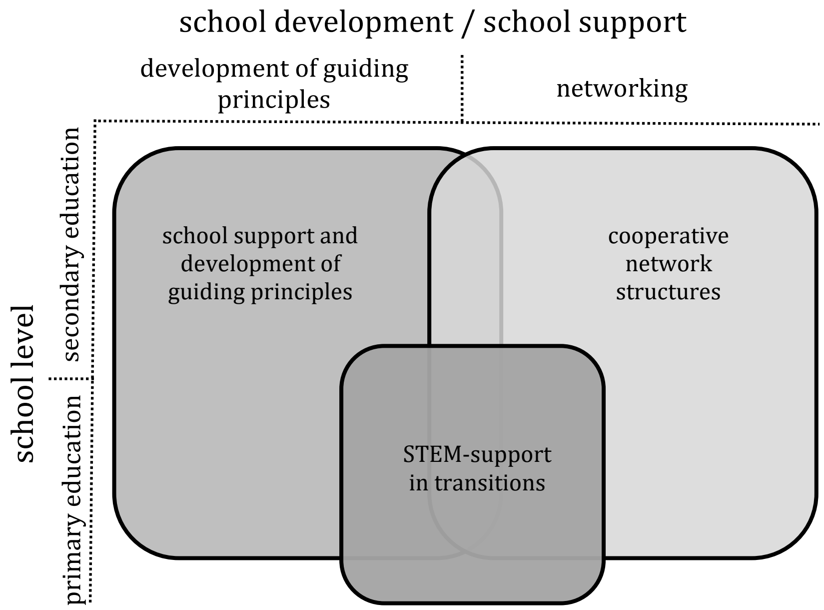

Figure 2: Structure of LemaS-module 1.

OUTLINE OF THE STRUCTURE OF THE INITIATIVE

The overall project comprises two core modules: Module 1 is dedicated to the development and evaluation of an achievement-supporting school culture from an educational science perspective (an overview of both the module's structure and subprojects is given by Figure 2). Module 2 focuses on the development and evaluation of diagnosis-based adaptive support concepts across the school subjects (see Figure 3 for its structure and subprojects). For the concrete implementation, 24 subprojects are planned across both modules, and three of those subprojects relate to mathematics.

One of the mathematics subprojects focuses on transitions between, e.g., elementary and primary education (located in module 1 in a 'STEM'-group; see Figure 2), another one focuses on the implementation of enrichment in regular classes, and a third subproject (in a group of seven subprojects of the 'STEM'-area) focuses on the diagnosis-based individual support of high-achieving and capable pupils in regular mathematics classrooms (each located in module 2; see Figure 3). Scientific supervision is provided by an interdisciplinary research network of 16 German universities (cf. DLR, 2018b; BMBF, 2018). All authors of this article are working in the mathematics subproject located in module 2 on diagnosis-based individual support. Therefore, a first insight into this subproject will be given, which simultaneously might provide an example of the 'types' and 'spirits' of concepts intended in module 2 considering existing national and international scientific findings.

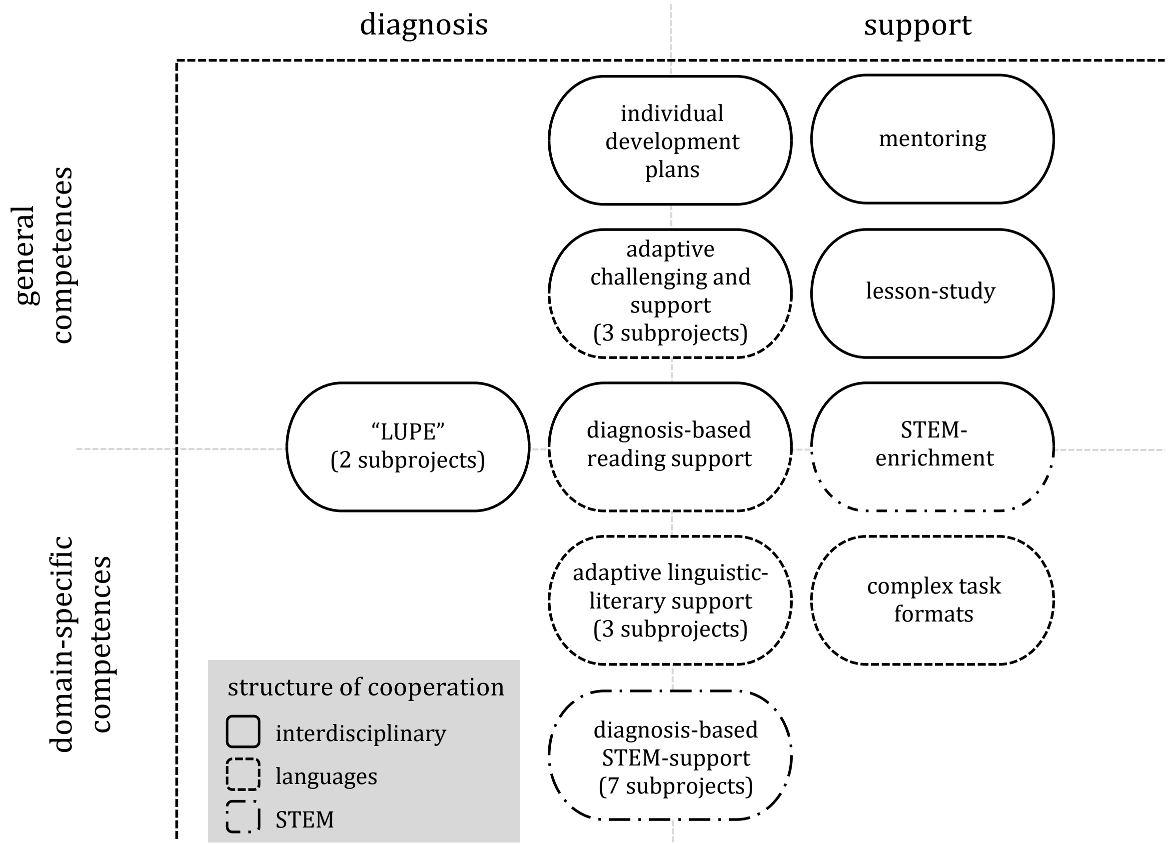

Figure 3: Structure of LemaS-module 2.

Following an application procedure, a total of 300 primary and secondary schools of all types were admitted to the initiative nationwide. In the first phase of the initiative between 2018 and 2022 scientists together with the participating schools in module 1 are going to develop a guiding principle for an achievement-supporting school culture that has a positive impact on the quality of teaching as well as on the motivation of both teachers and pupils. Additionally, cooperative network structures will be established. Simultaneously, the planned concepts will be developed in module 2. In a second phase between 2023 and 2027 the concepts will be evaluated and made available to other schools nationwide (DLR, 2018b; BMBF, 2018; Käpnick & Benölken, 2018).

AIMS OF THE OVERALL INITIATIVE AND THE STEM-SUBPROJECTS

According to the outlines given above, the overall aim of the initiative is to develop and to optimize both guiding principles and support concepts for high-achieving and capable pupils.

The subproject of module 2 on diagnosis-based adaptive support concepts, which is regarded as an exemplary focus in this article, is integrated into an overall network of STEM-didactics aiming at interdisciplinary exchanges (cf. Figure 3). Together with about 100 schools, concepts of the contents focused on are intended to be developed. As already indicated, adaptivity is an important purpose, i.e. the concepts should be developed in such a way that they can be flexibly adapted to specific local conditions of a certain school, and that they can be effectively used by teachers in their practice.

In accordance with the holistic-complex view of achievement development outlined above, the concepts of the STEM-network focus in the first phase of the initiative mainly on

- the development of a toolkit for learning process diagnostics, which consider subject-related competencies as well as intra- and interpersonal influencing factors that support or inhibit learning (e.g., individual cognitive or motivational characteristics, influences of parents or of extracurricular projects), and
- the development of learning arrangements (prototypes, tailored to different needs and potentials) for use in regular classes in STEM-subjects, which include in particular interest-driven, self-determined and research-based learning, as well as the use of digital media.

The concepts developed on the bases of scientific foundations will be documented and evaluated by teachers and the research team, and they will be continuously expanded and optimized. As a long-term aim, it is planned to publish the successfully tested concepts in the form of didactic-methodical handouts to offer them to a broad teaching staff.

PLANNED CONCEPTS IN THE MATHEMATICS SUBPROJECT ON DIAGNOSIS-BASED INDIVIDUAL SUPPORT

Against the background of both the holistic view on achievements' development and the equation of 'potential for excellence' and 'giftedness', the specific theoretical background of the subproject is given by 'mathematical giftedness' taking into account typical features like domain-specific criteria and personality traits or a dynamic development from an individual potential to high-above average performance influenced by inter- and intrapersonal catalysts (Fuchs & Käpnick, 2009).

Since the evaluation of the concepts to be developed is planned for the second phase of the initiative, we have to restrict ourselves to a first overview of the concepts, which are currently being developed in the first project phase as a result of our discussions with schools participating in the subproject focused on:

- Development of open, substantial problem fields (similar to approaches of, for example, Benölken, Berlinger & Veber, 2018; Nolte & Pamperien, 2017) and further formats for use in regular mathematics teaching.
- Development of in-school support concepts such as, for example resource rooms, research workshops, or concepts of 'mathematical discovery days' (in particular also in favor of realizing specific support in structurally weaker – e.g. rural – regions) and thematic orientations for mathematical research works (especially with regard to the support of pupils at secondary school age).
- Development of support concepts using digital media.
- Development of informal diagnostic tools focusing on mathematical giftedness for the use in regular mathematics classes (like indicator tasks or rating sheets of problem-solving procedures).
- Development of 'revolving door concepts' and optimization of such concepts already established at schools (for first impressions see Auhagen, 2019).
- Development of enrichment concepts and optimization of concepts already established at schools, such as support groups for mathematically interested and gifted pupils (e.g., modelled on the project 'Maths for little aces' established at the University of Münster; see Käpnick, 2008), also with special emphases such as specific support of girls (an example is given by Benölken, 2012).
- Development of concepts for teacher trainings, especially on the basis of the exploration of beliefs of mathematics teachers as to constructs of achievement or giftedness as well as to respective diagnostics and support.

PERSPECTIVES

The current state of work is that concepts have been agreed on with participating schools across all subprojects. The next and currently ongoing step is to develop the concepts locally at schools accompanied by scientists. Schools pursuing similar conceptual aims are connected by thematic networks, where an important focus as to the concepts' development is adaptivity, i.e. the possibility to transfer their cornerstones to other schools. The mathematics subproject located in module 2 reported above gives examples of concepts which are planned in other subprojects (and, thus, in other subjects) similarly. The initiative is currently still at an early development stage, but the concepts to be implemented in the mathematics subproject, for example, are expected to become more concrete in the course of the first phase as early as 2019. First informal evaluations of the single concepts will be conducted and they will provide a basis for continuous optimization.

As already indicated, a comprehensive evaluation of the concepts submitted at the end of the first phase is part of the initiative's second phase. Parallel to both phases, all planning and development processes will be documented in cooperation with a steering group, and the cooperation between schools and scientists will be evaluated from both perspectives.

The impressions gained so far indicate that both sides regard the cooperation as extremely constructive and that the specific expertise of both sides can enrich each other variously.

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SOCIO-MATHEMATICAL NORMS RELATED TO PROBLEM POSING IN A GIFTED CLASSROOM

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Abstract. *As a response to the calls for investigating norms and problem posing in gifted classrooms, this study aims to explore socio-mathematical norms related to problem posing in the micro-culture of a mathematics classroom of gifted students. A case study was conducted in a mathematics classroom with twelve students in a secondary school for gifted students. The data collection tools include an observation form, semi-structured interviews, and teacher's and students' notes. The main source of data consists of videos of twelve mathematics lessons. We observed three socio-mathematical norms related to problem posing (re-formulation of problems, generating new problems and sufficiency of the information in the problem) which were not reported in the literature. The paper discusses the implications of SMNs concerning problem posing and giftedness in mathematics.*

Key words: Problem posing; socio-mathematical norms; mathematically giftedness; mathematically talented students

INTRODUCTION

The importance of Problem Posing

Problem-posing, which is considered as an important intellectual activity in the process of scientific inquiry (Cai et. al., 2015), is also at the center of mathematical activity. Therefore, the development of problem posing ability is an essential target for mathematics education (Chen et. al., 2015). Researchers have begun to realize that the development of problem posing ability is as important as developing problem solving ability (Bonotto & Santo, 2015).

It is emphasized in the literature that more research is needed on how problem posing experiences influence students' mathematics learning through affective aspects (Cai et. al., 2015). Considering the complex nature of classrooms and the effect of classroom norms on learning, it is stated that there is a need for a theoretical framework and a detailed analysis to better understand how problem posing can be conducted in a classroom environment. More specifically, it is an important question of how teachers can shape classroom cultures to have a classroom environment in which students are involved in problem posing and how it becomes an accepted practice in the classroom (Cai et al., 2015).

Problem Posing in Gifted Classrooms

Problem posing is specifically important in the context of mathematically promising students (Singer et al., 2013; Xie & Masingila, 2017). There is research evidence of the effects of problem posing on conceptual understanding of mathematics, students' dispositions on mathematics, and creative mathematical thinking (Xie & Masingila, 2017). Problem posing is specifically crucial concerning mathematical creativity which is commonly described with its four dimensions: fluency, flexibility, originality, elaboration

(Mann et al., 2017; Leikin, 2011). In recent studies on the relationship between creativity and giftedness, it is seen that the researchers expanded their study areas to the dimension of attitudes and values based on the expected roles of the individuals with high potential in society (Singer et al., 2016).

Problem-posing refers to the generation of new problems and the re-formulation of the given problems. Therefore, problem-posing can occur before, during or after a solution to a problem (Silver, 1994). One of the problem posing strategies is manipulating a given condition and leaving the other conditions and the goal the same (Silver et. al., 1996).

"The nature of classroom culture and the role of the teacher in fostering mathematical expertise" is an emerging research area in the interdisciplinary field which includes mathematics education and gifted and talented education (Singer et al., 2016, p. 35). On the other hand, it is emphasized in the literature that there is a need for a theoretical framework to understand how problem posing can be conducted in a classroom environment (Cai et al., 2015). Considering the need for investigating classroom culture of gifted and talented students and the importance of problem posing within this culture, the aim of this study is to explore socio-mathematical norms related to problem posing in the micro-culture of a mathematics classroom of gifted and talented students.

Socio-Mathematical Norms

The notion of the socio-mathematical norm (SMN) is defined as the “normative aspects of mathematics discussions specific to students' mathematical activity” (Yackel & Cobb, 1996, p. 461). A norm describes the expectations and obligations that are negotiated among teachers and students (Yackel, 2004). Therefore, what becomes normative in a classroom is affected by the goals and beliefs of both teachers and students which also constrain what is acceptable as a mathematical activity in a classroom (Yackel & Cobb, 1996).

This paper will investigate the research question: “What kinds of socio-mathematical norms related to problem posing emerge in the micro-culture of a mathematics classroom of gifted and talented students? Moore-Russo and Weiss (2011) have raised the following questions regarding the existence of classroom norms in the context of problem posing: “Is it normative to encourage students to modify a problem (either to make it tractable, or to generate new avenues for exploration), or to introduce their own assumptions when solving problems? Do teachers commonly encourage students to pose their own problems?” (Moore-Russo & Weiss, 2011, p. 466). With regard to these questions, norms about problem posing in the literature include “(a) thinking of problem posing as a genuine and valuable mathematical activity; (b) agreements about what makes a problem (sufficiently) different from another one, why more challenging and/or more realistic problems are better, how problem posing and problem solving are related, etc. and (c) expectations of the role that students and teachers should play in the problem-posing activities” (Chen et al., 2015, p. 316).

METHODOLOGY

A case study was conducted to explore the research question above. The single case is a mathematics classroom with twelve students. This study was situated within a fifth-grade classroom in a private secondary school for mathematically gifted and talented students. Students (three girls and nine boys) are eleven years old.

The school selects its students using WISC-IV intelligence test.

The mathematics teacher of the classroom is a male mathematics teacher with an eight-year teaching experience. He was graduated from a four-year teacher education program in the Special Talent Education department and had been teaching the same class from the beginning of the first grade. The teacher follows the same mathematics curriculum and textbooks as in the mainstream schools but differentiates them to meet the needs of students.

The primary source of data consists of twelve mathematics lessons during 2018 autumn term. The qualitative data were analyzed using computer software. Identification of norms was based on the repetition of the events, and existence of student dimension as well as the teacher. The analysis framework was based on a list of 38 norms and descriptors of each norm which were specified considering the related literature. Each descriptor has a student dimension as well as a teacher dimension to be able to distinguish a two-way negotiation process. This current study will focus on the SMNs related to problem posing.

FINDINGS

Analysis of data indicated that problem posing is a common practice in the classroom's micro-culture and that students often generated problems as a homework activity. In terms of frequency, it was seen that norms related to problem posing were the most common norms among others in this class. A total of three SMNs were observed in the classroom. Below, observed SMNs will be explored with the teacher and student dimensions.

SMN1 (The problems given in the class are expected to be re-formulated)

The first SMN observed is "The problems given in the class are expected to be re-formulated." Transcriptions of lesson videos indicate both teacher and student perspectives. The excerpt below reveals that both the teacher and students initiate re-formulations of a problem.

1 T: Your friend is reading the question aloud…there is much to learn from each other.

2 S6: Zehra bought 187 meters of red fabric and 357 meters of white fabric.

3 One meter of fabric is 226 Turkish Liras (TL). How much has she paid in total?

4 S6: Teacher! Is it in Turkish Liras, Euros or Dollars?

5 T: Turkish liras first, but then I can ask you to convert it into dollars

6 T: Now just a second… what is 544...

7 S6: 544 meters of fabric

8 T: Well, here is the question. This is an enrichment question, listen carefully. What are 544 meters in decimeters? (How many decimeters are 544 meters?)

9 S5: 5440

10 T: Good answer. How many hectometers are 544 meters?

11 S5: 5,44 Five point forty-four

Although the teacher gave the currencies in Turkish Liras, it is seen that one of the students asked whether the result would be in Turkish Liras, in Dollars or Euros. This suggestion shows that the student has changed the problem condition by using the what-if-not strategy. Considering that the students are at grade 5, request for currency change might be considered as a re-formulation. It reflects the student dimension of this norm. As can be seen above, although S6 suggested to find out the solution in different currencies, the teacher did not follow it but suggested his enrichment question. The teacher re-formulated the problem (teacher dimension of the norm) and asked students to find the equivalent of the total fabric purchased in different length units (dm, hm). We considered changing the condition (length units) as an example of "what if not" strategy, in other words, a re-formulation.

SMN2 (Students are expected to generate new problems)

This SMN will be illustrated in two different instances. First, an excerpt from the class is presented below. In one of the lessons, the teacher wanted to show a video about the topic and one of the students paid attention to the geometric figures on the door in the video by generating questions about the door.

T: If the short edge of this region here is 2 cm and the long edge is 7 cm, what is the area?
S: 14
T: So what is the whole area? (There are four identical rectangular regions on the door)
S: If we multiply...
T: Fourteen times four is enough, no need to calculate. So what is the perimeter of this rectangle?
S: 18
T: Well suppose it is 16. If we assume that the long edge is 6 cm, the short edge is 2 cm, how many centimeters is the perimeter?
S: 16
T: What is the area of a square whose perimeter is equal to the perimeter of this rectangle? Or what is the length of an edge of the square?
S3: 16 square meters
T: Square centimeters
S3: Teacher, tell us the height, we'll find the cubic meter (volume).
T: Height or depth?
S3: Depth.

In the case of posing problems that started with the door scene appearing in the video, it is seen that the teacher generated problems in 2-dimensions. A student's suggestion to calculate the volume by thinking of the problem in 3-dimensions indicates the awareness of the student to create new problems in the classroom.

The second instance which is considered as an evidence of the SMN about generating new problems emerged during an interview with the teacher. The teacher reported that problem posing required a more tranquil environment and therefore he expected his students to generate problems as homework. He also mentioned that they published a booklet of problems they generated. One of the questions in the booklet is as follows:

> Delta Airlines has 850 aircraft. 60% of these planes fly abroad, and 38% of them fly inland. Consider that the airway company spends 100 liters on domestic flights and 275 liters on international flights. Five liters of fuel is 60 $, 2% of aircraft is in the care, and 1000 dollars are spent on each aircraft in maintenance. How much Turkish Liras does Delta Airlines spend? (Write the dollar exchange rate that you used)

SMN3 (Sufficiency of the information in the problem)

The third SMN observed is "Students are expected to decide whether the given information is sufficient to solve the problem (if the problem is wrong, it is expected to be detected)." Students, who were expected to find errors in problems, shared the errors they found with their peers. During the interview, the teacher stated that the problems that were found to be inaccurate were reported to the publisher of the booklet by the students who noticed the error. During the lessons, the teacher encouraged to detect faulty situations in the problems and to correct them. Students carried the faulty situations they encountered in the problems they solved individually at home to the class agenda and made attempts to correct the error in the conditions of the problem.

DISCUSSION AND CONCLUSION

As a response to the calls for investigating SMNs and problem posing in gifted classrooms (Singer et al. 2016; Cai et al., 2015; Moore-Russo & Weiss, 2011), this study investigated SMNs related to problem posing in a mathematics classroom for gifted students. As a result of the analysis of data, we explored three SMNs which were not reported in the literature.

Concerning SMN1 (re-formulation changing the given problem conditions), we question the level of re-formulations that come from students as well as teachers as a result of "what if not" strategy. Compared to the re-formulations in this study, literature reports different "what if not" strategies at a higher-level (Brown & Walter, 2005).

For the observed SMNs, there was evidence not only for teacher dimension but also for student dimension. In other words, students were aware of SMNs and sometimes took the initiative to suggest re-formulations or new problems as well as the teacher.

As described above, in some cases, students' suggestions for re-formulations were not followed by the teacher. However, the teacher asked his own problems. This might be a pedagogical decision of the teacher. Since the negotiation processes of norm construction might take time, the classroom culture should be observed for longer times.

This study has some implications for practice. Three SMNs related to problem posing were observed in twelve lessons, and this indicates the teacher's efforts and students' negotiations to establish such norms in the classroom. It should be noted that teachers and students did not have formal training on problem posing. Therefore, findings might be considered as a reflection of the existing classroom micro-culture.

It is possible that the teacher's background might be effective for his awareness of establishing a classroom culture where problem posing is valuable because notions such as fluency, flexibility, originality, and elaboration (Mann et al., 2017) are often the focus of attention in the curriculum of teacher education programs in Special Talent Education departments. Future studies could explore mathematics teachers with different backgrounds and their classroom cultures concerning SMNs related to problem posing.

Another implication is related to whether gifted and talented students should be integrated into the mainstream schools or should be educated in special classes (Leikin et al., 2017). Researchers should conduct comparative studies to compare the micro-culture of different mathematics classrooms and SMNs that are negotiated in there. Such studies have the potential to shed light on the discussions of how to meet the needs of gifted students.

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COLLECTIVE CREATIVITY IN MATHEMATICS: POSSIBLE SCENARIOS FOR SHARED MATHEMATICAL CREATIVITY

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Abstract. *Our purpose was to analyze the nature of shared creativity in mathematics in three different scenarios: working individually, working in groups without any mediation and working in groups with mediation of power in which the Creative Sharing Methodology was used. We selected 24 students from the 5th graders from a public school in the Brazilian, capital of Brazil. These students responded a test of creativity in mathematics composed of three versions used in each research scenario. It was observed that, in working individually, the teams presented fewer solutions, less varied and common ideas. Already in the collective work in which there was no intervention of the teacher, it was observed an improvement of performance. However, in the collective work with mediation of power, the teams produced less ideas, but with a greater level of originality. We conclude that shared creativity becomes more qualitative when power asymmetry is controlled.*

Key words: Mathematics Education. Shared Mathematical Creativity. Creative and Critical Thinking. Mathematical Motivation.

INTRODUCTION

The dominant literature in the area of creativity focuses on this phenomenon with emphasis on the individual characteristics, being less common the approaches that are dedicated to their collective aspects. Psychology, a field that dominated scientific studies of the phenomenon, tended to focus on the creative person, the creative context, the creative product, but found it very difficult to relate both aspects preferring to separate from each other and suspend them from historical time in that occur. In this way, these studies do not consider that “the person never thinks or acts outside of his intricate and dynamic system of social, material and institutional relations that constitute the human society” (Glăveanu, 2014).

However, new paradigms are being considered and some studies that bring the collective perspective of creativity can already be counted on. Emerging from these new ways of seeing creativity, scholars (Glăveanu, 2014; Sawyer, 2007) have advanced in the sense of opposing the psychological tendency to focus on the individual internal aspects of creativity, especially with an advanced look at the Systems Perspective proposed by Csikzentmihalyi (1996) and based on the cultural psychology of creativity (Glăveanu, 2014). For them, the brain / world dichotomy, which seeks to study the individual mind separating it from the outside world, is a misleading way of looking at human cognition. These authors consider the attempt to separate the inner (mind) from the external (social) unnecessary since these 'spaces' are co-constructive and permeable.

In the intersection between social, material and temporal, the phenomenon of creativity is configured as a complex network of interactions between people acting through tools and symbols at a given historical moment (Glăveanu, 2014). In judging by the hegemony of Capitalism on the planet, a system marked by individualism and competitiveness, we can witness the existence, in our current time, of asymmetric relations of power strongly influencing human interactions and assigning more or less active roles to people in the process of production of ideas, which also occurs in school spaces that are not on the margins of capitalist superstructures, but are strongly implicated in them.

Aware of the need to understand how collective creativity emerges from individual action situated in a social context in which people interact, we have been concerned with investigating how groups of children can produce mathematical ideas by solving problems in the team, which we have called shared creativity in mathematics (Carvalho & Gontijo, 2017). Furthermore, we seek to understand how the relations established in school settings configure the creative collective without neglecting to consider the historical moment in which emerged marked by individualism and strong asymmetry of power. In this perspective, we have studied how creative sharing occurs when children work in the school space a) individually, b) collectively without any form of mediation, and c) collectively with mediation of power. Therefore, considering that, even when working alone, these children bring traces of their life histories, coexist for months in interaction with the teacher and colleagues, and share cognitions and affections, we are interested in understanding how collective creativity is configured in these three forms of work with open mathematical problems.

In the present work, we will present quantitative results from a PhD research in progress that is investigating the shared creativity under a mixed methodology in which quantitative and qualitative data are being analyzed. The results brought here will allow us to answer the following questions:

a) Can creativity be a collective phenomenon? b) Are there differences between individual and collective work in the production of ideas? c) If so, what is the nature of this difference? d) Do power relations influence creative and collective work in classrooms?

Shared Creativity in Math

The social configuration that is being drawn in the West is pushing these countries to reconsider the dominant individualist view of creativity. The world is configured as a anything simple scenario in which "most teams operate in a more fluid, dynamic and complex environment than in the past" (Tannenbaum, 2012, p. 3). Thus, it is attributed to creativity a preponderant role in the search for survival in a world of uncertain future and intense change (Alencar & Fleith, 2003). Teamwork and creative ability therefore appear as demands for the scenarios in which people develop their work today.

We will use the term Shared Creativity to refer to the study of this phenomenon approaching it under the view of cognition shared with remarkable contributions of the field of distributed creativity. Originally, the shared cognition construct emerged in the context of organizational psychology research for over 20 years (Cannon-Bowers & Salas, 2001) as a benefit to the performance of teams and organizations. In this work, we consider shared cognition as "sharing and / or congruence of knowledge structures that may exist at different levels of conceptualization within a group and relate to the aspects of the group task" (Swaab et al., 2007, p. 188). Therefore, we assume shared creativity as a phenomenon that occurs in collectives in which people come together to perform some type of activity bringing their individual brands and contributing to the cognitive and affective sharing of their life experiences.

Power Relations

Emerging from a capitalist society structured by gigantic social inequality (OXFAM, 2017), today's school, and many decades ago, reproduces and reinforces asymmetrical power relations. In this sense, however simple that actions taken in school are (such as meritocratic reproduction that rewards some and exclude many or overvaluation of competitiveness), they can not be considered as actions disconnected from the history and social context in which it occurs. They should rather be considered as small mechanisms that make up a more global system of asymmetry of power that involves everything and everyone and that are replicated by students in their relationships.

As well exposed by Van Dijk (2015), we can not refer to power as an attribute of a particular person, but as the power of a "social position, being organized as a constituent part of an organization's power" (p. 21). In this sense, it is not a question of condemning the professionals of education, pointing them as reproducers of the asymmetric relations of power of the wider social context that chooses some as winners and condemns the others to failure. One can not concentrate "the guilt on a teacher or a biased passage of a textbook, since the form of influence can be much more diffuse, complex, global, contradictory, systematic and almost unnoticed by all involved" (Van Dijk, 2015, p.22). It is really a matter of clarifying that the differential can be in the possibility of the teacher, by using his sensitivity and critical sense, to perceive himself as someone who can break with this segregating logic.

METHOD

Twenty-four children from 5th ground, all 10 years of age and enrolled in a public school in the Brazilian, capital of Brazil, participated in the study. They were divided by the teacher into 8 groups of 3 children each, seeking to form heterogeneous teams in relation to mathematical development. The students responded to a test of mathematical creativity composed of 3 versions with 3 items each, one being answered individually, another answered in trios without any mediation and the last one performed in a power mediation context in which the Creative Sharing Methodology was applied (CSM).

In the CSM, the students' work is directed, in each test item, through 4 stages: Individual Production (students present solutions individually), Evaluation to Blindness (each component of the trio corrects the solutions of the others without being identified), Negotiation (each team member defends their ideas, suggests modifications to their peers, and combines ideas by producing a new one) and Compilation (discuss which ideas will compose the answer sheet).

In each trial version, there are two open-ended problems items and one about problem-elaboration. In the first version, the creativity score is calculated by the mean of the individual scores of the components of each team. In the other versions of the test, the scores are calculated by the solutions presented by the team in the answer sheet of each item. We analyzed the performance evaluating the evolution of the Final Creativity scores and each criterion separately: Fluency, Flexibility and Originality.

RESULTS AND DISCUSSIONS

In Table 1, we noticed that there was an increase of score in all groups when comparing the notes of the first and second versions and the notes of the first and third versions of the test. In the comparison of the second and third version, we noticed that only group 6 presented a decrease in the creativity score.

Team	Average Version 1	Average Version 2	Average Version 3	Growth Rate 1	Growth Rate 2	Growth Rate 3
G1	1,40	2,13	2,25	52%	61%	6%
G2	1,20	1,69	1,74	41%	45%	3%
G3	1,24	1,68	1,83	35%	47%	9%
G4	1,50	1,84	1,98	23%	32%	8%
G5	1,61	2,35	2,67	46%	66%	14%
G6	1,47	1,88	1,67	28%	14%	- 11%
G7	1,41	1,74	1,92	23%	36%	10%
G8	1,66	1,95	2,18	17%	31%	12%
Total	1,44	1,90	2,03	32%	41%	7%
SD	0,16	0,23	0,32			

Table 1: Creativity scores on three versions of the test[8]

Using the scores of the three versions, we compared the averages to find out if there was a development, since we saw that the great majority of the groups presented progression in the performance, that is, increase of the scores during the realization of the three versions of the test. Attested to the normality of the data, we opted for the paired T test in order to compare the averages and verify if there was improvement in the performances. Thus, the T test showed that, at a significance level of 5%, there was a significant difference in performance between the scores of version 1 and version 2 of the TCM test ($p < 0.000$) and between the scores of version 1 and version 3 of the TCM test ($p < 0.000$). Regarding the comparison between the scores of version 2 and version 3 of the TCM, the T test shows that, at a significance level of 5%, there was no significant difference between the means ($p < 0.058$) due to this value is one slightly higher than expected.

In others word, the data show that there was a significant improvement in the performance of the groups when they moved from individual work to collective work without mediation of power. They also show that this performance was superior when comparing the scores obtained in the individual work with the collective work with mediation of power. However, when performance was analyzed in the two forms of collective work (without and with power mediation), this performance improvement is not statistically significant when considering a significance level of 5%.

In order to understand this data more deeply, we analyzed the results in each criterion evaluated in the test (fluency, flexibility and originality). Thus, we noticed that there was an increase in scores when comparing version 1 with the two collective versions of the test. However, the same can not be observed with regard to the comparison between the scores of version 2 and version 3.do TCM.

Regarding Fluency, there were significant differences, at a significance level of 5%, in Total Fluency scores. In the comparison between versions 1 and 2 ($p < 0.001$) and between versions 1 and 3 ($p < 0.041$), the difference was significant and positive, indicating that there was an increase in the number of responses given in the collective versions of the

[8] Growth rate 1: Percentage of the difference between the scores of the first and second versions.
Growth rate 2: Percentage of the difference between the scores of the first and third versions.
Growth rate 3: Percentage of the difference between the scores of the second and third versions.

test. On the other hand, it can be seen that, in the comparison between the collective versions 2 and 3, although it was statistically significant, this difference was negative, that is, the students had fewer ideas in version 3.

In terms of Flexibility, there are also significant and positive differences when comparing the means of versions 1 and 2 ($p < 0.006$) and between versions 1 and 3 ($p < 0.039$) of the test, ie the ideas produced collectively were more varied than those made individually. However, there was no significant difference in performance when comparing the flexibility scores of version 2 and 3 ($p < 0.37$) of the test.

On the other hand, the data show that in all the situations there was a significant improvement with respect to the Originality scores. Thus, we can observe that, at the 5% significance level, there was a significant improvement in performance when comparing the originality scores of the three versions of the test, either comparing version 1 with version 2 ($p < 0.003$), version 1 with version 3 ($p < 0.000$) or version 2 with version 3 ($p < 0.001$).

In this sense, it can be considered that the students were more original in the version of the collective test with mediation than in the other versions. Thus, it is observed that there was development in terms of quantitative and qualitative aspects of creativity in mathematics when the students leave the work isolated and begin to work collectively. On the other hand, when comparing the two collective versions, one can notice the development of more qualitative aspects (originality), since the students presented less quantity and less categories of solutions, but these solutions were more original.

CONSIDERATIONS

We can conclude that, during the stages of solving open-ended problems, it was shown that individual work was less conducive to the production of ideas insofar as they were in smaller quantity and at lower levels of quality. Although it can not be considered a completely individual action and taking into account that "the mind emerges in the joint mediated activity of people" (COLE, 1996, p 104) we can say that this form of work occurs in an isolated way but not solitary given that we resorting to what was shared with us during moments of interaction to think and act. And this isolation proved less appropriate for the quality gain of the ideas produced than the collective work. On the other hand, collective work, even without any mediation of power, allowed students to exchange information and enrich the process of creative sharing. In the third stage, where there was a methodology of power mediation, the ideas produced were shown in less quantity, but, in compensation, there was the qualitative gain of these solutions. In this way, we can answer our questions as follows: creativity can be a collective phenomenon and just in the social environment does it comes to life, with qualitative differences when this work occurs in teams mediated by the teaching action.

This demonstrates that when they have had the opportunity to dialogue democratically, with openness to produce ideas, to evaluate the work of colleagues and to negotiate the solutions produced, the students have been able to add more criteria to their work, inspired by the productions of their peers, allowing themselves to choose those solutions that they found most promising for qualitatively creative work. Thus, in the collective work in which the voices could be equaled, the development of higher order thinking (PITTA-PANTAZI and SOPHOCLEOUS, 2017) was observed in which the students could reflect, act with criticality judging the ideas produced (and not the producers of them) and being able to combine solutions presenting unusual ideas. Finally, it is necessary to demonstrate that the teaching activity proved important in that it allowed the class a democratic way of expressing ideas.

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AN INITIAL INVESTIGATION INTO TEACHER ACTIONS THAT SPECIFICALLY FOSTER MATHEMATICAL CREATIVITY

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Abstract*: While mathematicians and mathematics educators agree that students should be exposed to the creativity inherent in mathematics, there still is a need for further research showing how this can be done at the tertiary level mathematics. This report uses empirical evidence in conjunction with Sriraman's Five Principles for maximizing creativity framework to explicate teaching practices that can foster mathematical creativity in the classroom. The report provides a practical guide for mathematics teachers who would like to value and nurture creative mindsets in their students.*

Key words: teaching practices, mathematical creativity, tertiary-level

INTRODUCTION

Mathematical creativity seems to be an important part of mathematics (Hadamard, 1945), and more recently, mathematics education (Schumacher & Siegel, 2015). There are researchers that have studied pedagogical actions of fostering mathematical creativity at the K-12 level (e.g., Levenson, 2011) and in the tertiary level (e.g., Zazkis & Holton, 2009). We believe a more in-depth investigation, a theoretical backing (Sriraman, 2005), and verification will add to our understanding of ways to foster mathematical creativity.

In this report, we take the five principles conjectured by Sriraman (2005) and expand them to twenty actionable items. Then, using data from an inquiry-based learning tertiary classroom on introducing proofs, we offer student testimonials that they were creative and why they felt creative. These explanations are analyzed using the twenty teacher actions. We conclude the proposal with new possibilities for future research.

THEORETICAL FRAMEWORK

We view mathematical creativity as a process of offering new solutions or ideas that are unexpected for the student, with respect to their mathematics background or the problems they have seen before (Savic et al., 2017a). A heavy influence of our definition came from Liljedahl and Sriraman's (2006) discussion on mathematical creativity and its constructs in the classroom. Focus on the creative "process" is one of four major theoretical perspectives in researching creativity: the viewpoint of the *person*, the *product* that arises, the *process* by which that product is created, and the *press* or the response that the product elicits from others (Rhodes, 1961). It is difficult to define what "new solutions or ideas" and "unexpected" are; therefore, with our definition, the students designate what "new" is, grounding originality within "the student" and their background. Vygotsky (1984; as cited by Leikin, 2009) stated that there is significance in both *relative* (how we employ the definition) and *absolute* creativity, with absolute being discoveries at a global level. This is similar to the Big-C and little-c creativity discussed by many researchers (e.g., Beghetto & Kaufman, 2007). Focus on mathematical creativity instead of creativity in general relies on the notion that there are significant differences of creativity between domains (Baer, 1998).

This student-centered, process-oriented perspective yields a central question: **how does one foster such mathematical creativity?** Zazkis and Holton (2009, pp. 359-360) described both problems and creativity-fostering actions, including multiple solution tasks (Leikin, 2009), learner-generated examples, open-ended problems (Zaslavsky, 1995), and creating new mathematical definitions. We add to this literature by demonstrating creativity-fostering actions in one classroom derived from Sriraman's principles below.

Sriraman's Five Principles for Maximizing Creativity

Sriraman (2005) conjectured five principles for maximizing creativity in mathematics: gestalt, aesthetic, free-market, scholarly, and uncertainty. We say "conjectured" since these were recommendations according to Sriraman and have been minimally investigated in the classroom (Savic et al., 2017b).

The *Gestalt principle* is based off of Gestalt psychology and Wallas' four-stage creative problem-solving process: preparation, incubation, illumination, and verification (Wallas, 1926; Hadamard, 1945). This principle requires that instructors allow students "to engage in suitably challenging problems over a protracted time period, thereby creating the opportunities for the discovery of an insight and to experience the euphoria of the "Aha!" moment" (p. 27). The *aesthetic principle* looks at the beauty of a mathematical process or solution. Characteristics such as elegance, efficiency, atypical, and combination of disparate ideas are part of the aesthetic principle. The *free market principle* revolves around taking risks when presenting a solution. Sriraman explains: "Professional mathematicians… take a huge risk when they announce a proof…The implication… for the classroom is that teachers should encourage students to take risks…allowing them to gain experience at defending their ideas upon scrutiny from their peers" (p. 28). The *scholarly principle* looks at creating an environment where "[teachers] should be flexible and open to alternative student approaches to problems… nurture a classroom environment in which students are encouraged to debate and question the validity of both the teachers', as well as other students', approaches to problems… (p. 28). The *uncertainty principle* is based on the idea that mathematics as a discipline involves uncertainty and we should expose our students to that concept by appealing to the history of mathematics and showing that many problems to years to solve.

TEACHER ACTIONS THAT FOSTER CREATIVITY

The data presented in this report were collected in Spring 2016 from students in an introduction to proof course at an institution in the Southwestern United States. To explicitly value creativity in the classroom, the instructor, Dr. Eme, used the Creativity-in-Progress Rubric (CPR) on Proving (Savic et al., 2017) while implementing an Inquiry-Based Learning (IBL) teaching pedagogy. Fourteen students were invited to participate in interviews at the end of the semester and seven participated. The interviews were conducted by a member of the research group and were from 15-75 minutes in length.

Two members of the research team coded the interviews separately and then met to discuss their codes and consensus on codes was reached. For the majority of the coding process, the two members used leading questions based on the 5 principles to code the interviews. For example, to code for the Free Market principle, we looked through the interviews for answers to "Did the course or instructor's actions/teaching promote students to take risks while presenting solutions? Did the course or instructor's actions/teaching create a safe environment for students to take risks?'' The following student quote was coded under the Free Market principle as it answered these questions:

> I think really the structure of the course is what helped to expand on my mathematical creativity when I thought I didn't have any. So, um, and you know the structure of the course meaning, you know the group discussions, the group talks, um the presentations were a pretty big deal.

Based on student answers to the question "Did you feel creative in this course?" we found that students reported feeling very creative in the course as well as recognizing and valuing the creativity of other students. For example, two of the interviewees' responses were:

> In regard to mathematics, I think I am on the spectrum that generally believes there's no need for creativity in mathematics. That's been a key reason why I enjoy math. I know if I get the answer then I have done it correct. There's a set process and if I learn the process then I will be successful. However, this class especially has proven to me how untrue that belief is.

> …working with…trapezoidal numbers, and once we saw the different representation of consecutive numbers, you know minus 1 and plus 1 versus plus 1, plus 2, which was all, the entire class' first initial connection was, you know plus 1 and plus 2. When he flipped it to the other side, everyone was just 'Wowww! That's so amazing!' And then but, then we went on and worked with trapezoidal numbers a little bit more, and everybody's making that connection.

This led to our research question: What specific teacher actions that contributed to the feeling of creativity amongst the participants in the classroom community? Dr. Eme seemed to use various teacher moves in her classroom, and we have triangulated moves with instructor journals and interview data as well as students' interviews.

One student stated that Dr. Eme valued their contributions without passing judgement.

> so I think when she … gave us like that reflection of like what it means to be creative, we kind of, she kind of just like told us like 'No proof is gonna be exactly the same. Like none of your proofs are actually gonna be the same as each other and you guys are all gonna come up with different ideas'. And she kind of like helped us, like she never like hindered those ideas. She was like 'Oh, well maybe it can work like this. Maybe it can work like that. You just have to like see'.

This teacher action aligns with the Scholarly principle from Sriraman as students were encouraged to present their work to the instructor or other students and other students were encouraged to build on that work.

Another student spoke to the fact that Dr. Eme allowed for multiple attempts on problems.

> I think also the feedback that Dr. [Eme] would give us on our homework. Cause we would turn it in and we would be able to have multiple submissions of our homework to make sure that we would get the proof right.

Allowing students to turn in proofs more than once encourages students to take risks as outlined in Sriraman's Free Market principle. This student is also telling us that the instructor was allowing students to try an approach and fail without penalty. Interestingly, this quote also reflects the instructor implementing the Gestalt principle by allowing freedom of time and movement, giving students a chance to reach that AHA moment.

Another interviewee spoke to the instructor's style of giving guidance, but not answers to student questions.

> I think a lot of it was the way the class was structured and Professor [Eme] gets a lot of credit for that. She very much threw us in there and said 'sink or swim'. And you know it was 'I'm here if you need a little guidance but you're never gonna get an answer from me, so don't even bother asking for an answer; you know it's not about the answer it's about the process.

Dr. Eme allowed students to experience the difficulty and uncertainty involved in doing mathematics. Therefore, she exposed students to the authentic practice of being a mathematician, where a path to solving a problem may not be clearly defined or may not even exist.

The next student quote demonstrated the instructor implementing the Aesthetic principle and encouraging students to see the beauty in mathematics.

> There's one guy in particular who had a way of coming up with these tricks that just made proofs very efficient. Instead of having ten lines, he would have three and it would be fully proved. And it was really neat…..It was wonderful watching his work.

Through encouraging students to see each other's work (which could be considered an implementation of the Scholarly principle), students would judge other students work, both for correctness and form, but also for aesthetic appeal. This process is expanded upon by another student interviewee reflecting on the work of a student she labeled as particularly creative.

> So his creative moment, I could then use to expand on and do something a little different with to have my own creative moment. And then I could show that to the class and then you know somebody else in class could pick that up and manipulate it for a different proof and do other things with it. And so, we were doing these things that are not the road most travelled I guess and then … those become an integral part of the road we are traveling together, and yet each time we're changing it to be what we need it to be and expanding on it and having our own creative moments, based on a creative moment that somebody else had before us.

Teacher Actions Extracted from Sriraman's Five Principles

The above quotes speak to teacher moves that the students were experiencing in the classroom that encouraged their creativity. These teacher actions have a high alignment with Sriraman's Five Principles. In fact, based on our attempts to explicate the actions encapsulated in Sriraman's (2005) Five Principles, we enumerate below twenty specific teacher actions that can maximize student creativity. Several of these teacher actions are the same as we showed in the last section based on the student quotes, however we discuss additional actions here that were not observed in this classroom yet are highlighted in the Five Principles. The teacher actions outlined here are more detailed than the Five Principles themselves and displaying them in this way makes them accessible to be implemented by any instructor wanting to foster creativity in their classroom.

The Gestalt principle contains three specific teacher actions within it. Following this principle, a teacher should:

- allow for freedom of time and movement;
- discuss explicitly that time, effort, and energy are needed to solve problems;
- assign challenging problems and tasks.

Allowing for freedom of time and movement may incorporate classroom practices such as giving flexible due dates to allow time to really work through a problem, allowing revisions of problems, and encouraging different approaches to problems.

Four relevant teacher actions are covered by the aesthetic principle. When applicable to the classroom situation, a teacher should:

- point out the elegance/novelty/beauty of certain solutions/approaches;
- point out connections between disparate ideas in problem solving;
- point out any atypical thinking/solutions;
- point out simple solutions to complex problems.

Our study shows that the free market and scholarly principles have a lot of overlap, thus we present nine teacher actions that relate to one or both of these principles. To enact the free market and scholarly principles, a teacher should:

- encourage students to present their solutions and approaches;
- encourage students to defend their solutions and approaches;
- value students' contributions;
- not penalize students for trying a different approach and failing;
- encourage students to debate and discuss the teacher's approaches and the other students' approaches/presentations;
- elaborate on how these discussions contribute to the process of knowledge building;
- point out when a student builds on the work of another student;
- encourage students to make generalizations;
- allow students to problem pose.

Finally, there are four teacher actions embodied in the uncertainty principle. For this principle, a teacher should:

- point out the difficulty and uncertainty of doing mathematics when students are working on challenging tasks;
- provide affective support to students when they experience frustrations;
- encourage perseverance;
- expose students periodically to examples from history to explain that certain concepts took years/centuries to develop.

DISCUSSION

The student interview data in this study provided evidence that students felt creative in this classroom and were also able to value creativity in other students. Additionally, students were able to identify specific practices of the instructor that contributed to these feelings. These teacher practices along with the ones extracted from Sriraman's Five Principles provide a robust framework for a classroom that fosters creativity. Making these teaching practices explicit from the Five Principles allows instructors that want to encourage creativity in their classroom to "pick from the menu" of practices and implement them.

These twenty teacher actions also present several open questions that need future research. The classroom discussed in this report did not incorporate all twenty teacher actions, yet still was successful in fostering student creativity. Therefore, which teacher actions are the most or least important in promoting a creativity-focused learning environment? Is there some minimal spanning set of teacher actions? That is, is there a least number of practices that one could implement and still see similar results to the classroom presented here?

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COMPARISON OF GIFTED AND MAINSTREAM 9TH GRADE STUDENTS' STATISTICAL REASONING TYPES

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Abstract *There have been studies investigating gifted students thinking skills while they attempt to solve a challenging problem (Leikin, 2011) but only a few studies specifically focused on gifted students' statistical reasoning abilities. Garfield and Chance (2000) defined statistical reasoning as "the way people reason with statistical ideas and make sense of statistical information" (p.101). In this study, statistical reasoning types of 9th grade students attending school for gifted (n_1=49) or school for mainstream education (n_2=42) were compared. Turkish version of Statistical Reasoning Assessment (Karatoprak, 2014) instrument was used to investigate whether there are any differences between these two groups of students' statistical reasoning types. Descriptive results of 8 subscales of the test and results of Mann-Whitney-U Test for these subscales are reported.*

Keywords: *gifted, statistics, statistical reasoning, mathematics.*

INTRODUCTION

Giftedness is a rare resource for any nation and a nation's success is considered to be depending on how its gifted students are treated and educated (Porter, 1990). However, this education is more difficult than anticipated because gifted students need more individualized and intensive curriculum, more challenging tasks, opportunities for creative expression and enrichment activities (Ford, Russo & Russo, 1993). To provide such kinds of appropriate educational environments that flourish gifted students' abilities, first, their distinctive thinking capabilities should be uncovered. Yet, giftedness is a domain specific phenomenon emerging under favourable conditions (Renzulli, 2002). As a specific type of gifted students, mathematically gifted students are able to express problems in mathematical ways and making generalizations for mathematical patterns, being flexible and creative in their problem-solving strategies, and being fluent in mathematical skills (Hekimoğlu, 2004). In addition to mathematical giftedness, mathematical reasoning includes a variety of abilities such as constructing and testing a mathematical hypothesis, discerning mathematical patterns, as well as applying a broad range of mathematical problem-solving strategies (Kramarski & Mevarech, 2003). On the other hand, despite the existing consensus on mathematical thinking abilities of the gifted students, whether this can be expanded to statistics[9] is still on debate.

To highlight the importance of statistics education, National Council of Teachers of Mathematics (2013) stated that statistics education deserves more attention because it helps us to make sense of the surroundings in a more logical way by providing daily information. However, students might be misguided by calculating some statistical measures without meaningful interpretation of data. At this point, statistical reasoning

[9] The view of statistics as a branch of mathematics has been replaced by the understanding of statistics and mathematics as separate disciplines (Garfield & Chance, 2000).

plays a pivotal role to look behind the scene by interpreting the numbers in the right way. According to Garfield and Chance (2000), statistical reasoning involves reasoning with statistical ideas and interpreting statistical information.

Although many studies were conducted to investigate mathematical giftedness (Krutetskii,1976; Sheffield,1990; Hekimoğlu, 2004; Heinze,2005) and mathematical creativity (Sririman,2005; Sak &Maker, 2006; Sheffield, 2009; Leikin, 2009; Leikin, Koichu & Berman, 2009; Bahar & Maker, 2011) as well as gifted students' mathematical reasoning (Berg & McDonald 2018), research on gifted students' statistics education is scarce. This is an expected consequence curricular focus on arithmetic and algebra which are recognized as the base of mathematical studies.

Statistical reasoning should be included in gifted education and pertinent research areas because gifted students should make sense of statistics to support our information-laden world. Additionally, it should be added here as well that the comparative studies of mathematical gifted students' and mainstream students' cognitive functions are limited (Berg & McDonald 2018). Hence, there is a need for comparative studies in specific domains including statistics to highlight the relatively superior abilities of the gifted students, thus appropriate differentiated learning environments might be designed. Based on the discussion provided above, this study is designed as an exploratory study to investigate differences between statistical reasoning types of gifted students and mainstream students.

LITERATURE REVIEW

In this section, the definition of giftedness, mathematical giftedness and statistics education are covered based on the pertinent literature.

Giftedness

To understand what giftedness is, one needs to consider commonly used and widely accepted definitions of giftedness in historical order. One of the pioneering persons in gifted education, Terman (1928), defined giftedness as "the top 1% level in general intellectual ability, as measured by the Stanford-Binet Intelligence Scale or comparable instrument" (cited by Renzulli, 2002). Then, contemporary researchers highlighted the multifaceted dimension of giftedness. In 1978, Joseph Renzulli introduced three ring theory, in 1983 Howard Gardner used the term intelligence instead of talent and introduced seven multiple intelligences, and Sternberg (1985) constructed his triarchic theory of intelligence (Sternberg, 1985). Renzulli (1978) built his theory on gifted behaviours as opposed to treating individuals as gifted individuals. According to his theory, a gifted behaviour is composed of three components; (a) above-average ability, (b) high levels of task commitment, and (c) high levels of creativity. In 1983, Gardner defined intelligence as the capacity to solve problems or create products which are valued in one or more settings (Gardner & Hatch, 1989). According to Gardner (2003) intelligence is; "*a property of all human beings, a dimension on which human beings differ, and the way in which one carries out a task in virtue of one's goals"* (p.8). In following years, Sternberg (1985) defined human intelligence as "mental activity directed toward purposive adaptation to, selection and shaping of, real-world environments relevant to one's life" (cited by Renzulli, 2002, p.120). Based on the definition of human intelligence, Sternberg developed his triarchic theory of intelligence, arguing with reliance on IQ tests as the sole determinant of intelligence. In his theory, intelligence can be divided into three sub-categories which are analytical, synthetic/creative, and practical (cited by Renzulli, 2002). In later years, the domain-specific definitions of giftedness have been emerged due to existing differences among gifted pupils. As domain-specific giftedness, mathematically giftedness has been identified and studied in a variety of research (Krutetskii, 1976; Sowell, Zeigler, Bergwall & Cartwright, 1990; Heinze, 2005).

Mathematical Giftedness

Mathematical giftedness can be defined as being able to do mathematics typically accomplished by older students or to engage in qualitatively different mathematical thinking process than mainstream counterparts or chronological peers (Sowell, Zeigler, Bergwall & Cartwright, 1990). Similarly, Heinze (2005) argued that mathematically gifted students have high ability to verbalize and explain the solutions, and to use the insight in the mathematical structure of a problem in order to solve it by deducing or calculating the solution compared to their non-gifted counterparts. Additionally, in a study about the factors that affect the mathematical reasoning of mathematically gifted and typically achieving students, visual spatial working memory has been found a prominent factor in this regard (Berg & McDonald 2018). In addition to comparative studies in the mathematical giftedness, Karabey (2010) focused on the relationships across different concepts of giftedness and found that mathematical giftedness positively related to creative problem-solving and critical thinking.

The literature suggests that mathematical ability is generally manifested in accomplishing tasks related to the mathematics curriculum (Koshy, Ernest & Casey, 2009). Therefore, the mathematical abilities of gifted students in different fields of mathematics such as statistics should be investigated.

Statistics Education

Garfield (2003) stated the primary objective of statistics education as enabling students to produce reasoned descriptions, judgments, inferences, and opinions about data. On the contrary, in most countries, statistics education still does not go beyond drawing graphs or calculating some statistical indices like average and standard deviation. Besides, there is no satisfactory focus on data collection, critical thinking, interpretation and prediction in educational strategies (Koparan & Akıncı, 2015). In this regard, statistical reasoning that is aimed inherently in the statistics education plays a pivotal role to make sense of statistical indices.

Garfield and Chance (2000) defined statistical reasoning as “the way people reason with statistical ideas and make sense of statistical information” (p.101). Also, statistical reasoning includes making interpretations based on the data set. After they defined statistical reasoning, they identified the characteristics of statistically reasonable students as follows; combining data and chance factors which lead them to make reasonable statistical judgements; identifying a graph from a description of a variable; matching two versions (graphs) of the same data; understanding the impact of adding or removing value(s) and understanding the reasons for shape. Existing studies on statistical reasoning were not focusing on gifted students thinking. Using a statistical reasoning instrument developed and test for general purpose may shed some light on how gifted students’ statistical reasoning is differs from mainstream students.

METHODOLOGY

The purpose of the study was to compare gifted students statistical reasoning types with mainstream students; thus two high schools were chosen and all of 9th grade students of these schools were administered the test. The first author was a 9th grade mathematics teacher of both of these private high schools. The school for gifted students (*School G*) has chosen its students at 5th grade by administering a battery of tests (WISC-R IV and TTCT). The students were identified as gifted students yet none of the was not labeled as mathematically gifted. There were n_1=49 gifted male students from School G in this study.

For comparison, another private high school (*School M*) was chosen from Istanbul. There were two reasons for choosing this school. Firstly, in order to control the influence of instruction (mainly statistics teaching) a high school that same mathematics teacher is working was chosen. Secondly, School M accepts students from 20 percent from the High School Selection Exam, (a nationwide exam administered to 8th grades for entering high school). This high school selection test is very competitive. School M has only managed to take students from the first 20 per cent which is not so high percentage. There were n_2=42 (21 male and 21 female) mainstream students participating in this study.

During 2017-2018 academic year, the researchers administered the statistical reasoning test, Statistical Reasoning Assessment (SRA) which consist of 20 multiple choice statistics and probability questions, to the participants. In the test, most responses are followed by a statement of reasoning and/or explanation. Students select the answer which fit well to their thinking strategies for each problem. Statistical Reasoning Test which is developed by Garfield (2003) was translated and adapted into the Turkish context by Karatoprak (2014). The choices of the instrument were designed to identify not only correct reasoning types but also misconceptions. While each right choice signals a correct reasoning skill, each wrong choice signals a specific misconception. Thus, the instrument diagnoses 8 correct reasoning types and 8 misconceptions. The Turkish version of SRA was used in this study to compare statistical reasoning types of the gifted and mainstream students. The test took approximately 35 minutes to completed by both types of participants. The research design was causal-comparative since the situation, which is statistical reasoning types of 9^{th}graders, in this case, was aimed to be described with quantitative data and to determine existing differences between gifted and mainstream students (Creswell, 2012). The groups of gifted and mainstream are already established, and the purpose of the study is to explore the difference between the groups in terms of statistical reasoning so this study can be considered as causal-comparative design. To analyse the data Statistical Package for the Social Sciences (SPSS) 24 was used and due to the type of the data, while comparing the group results for each subscale independent sample t-test or Mann Whitney U Test was administered.

RESULTS

Results of each subscale (8 correct reasoning subscales and 8 misconception subscales) are presented with descriptive statistics which are provided for both groups separately. In order to examine whether there exists a difference between subscale scores of gifted students (N1=49) and mainstream students (N2=42) firstly Kolmogorov-Smirnov normality test was administered for 16 subscales. Significant value of normality test for each of the remaining was found .00 in both groups. Thus, normal distribution assumption was failed for each subscale scores of both groups. Therefore, the results of the groups were compared with Mann-Whitney U Test. However, for two subscales, distinguishes between correlation and causation subscale, were constituted by just one dichotomous question, thus chi-square test was used to examine differences between the scores of the groups for these subscales. The below table displays each groups' correct reasoning subscales scores.

Correct Reasoning Subscales	Items	Gifted Students' Mean Scores	Mainstream Students' Mean Scores	Statistically Significant or Not Significant?
Correctly Interpret Probabilities	2, 3	0.800	0.666	not statistically significant (p; 0.115>.05)
Understand how to select an appropriate average	**1, 4, 17**	**0.521**	**0.242**	**statistically significant (p; .001<.05)**
Correctly computes probability	8, 13, 18, 19, 20	1.050	0.993	not statistically significant (p; .735>.05)
Understand Independence	**9, 10, 11**	**1.000**	**0.700**	**statistically significant (p; .017<.05)**
Understand Sampling variability	14, 15	0.233	0.212	not statistically significant (p; .852<.50)
Distinguish between causation and correlation	16	0.550	0.466	not statistically significant (p; .232>.05)
Correctly interpret two-way tables	**5**	**0.775**	**0.400**	**statistically significant (p; .002<.05)**
Understand importance of large samples	**6, 7, 12**	**0.696**	**0.497**	**statistically significant (p; .038<.05)**

Significant results are shown as bold.

Table 1: The results for correct reasoning types

Out of 8 correct statistical reasoning subscales, gifted students' scores were statistically significant for 4 subscales. These results are consistent with the idea of gifted students' mathematical reasoning to be more complex than mainstream students' mathematical reasoning (Piburn & Enyeart, 1985).

Misconception Subscales	Items	Gifted Students' Mean Scores	Mainstream Students' Mean Scores	Statistically Significant or Not Significant?
Misconceptions involving averages	1, 4, 15, 17	1.140	1.240	not statistically significant (p; 532<.50).
Outcome orient	2, 3, 11, 12, 13	0.249	0.186	not statistically significant (p; .287>.05)
Good samples have to represent a high percentage of the population	7, 16	0.550	0.472	not statistically significant (p;.631>.05)
Law of small number	12, 14	0.325	0.283	not statistically significant (p; .441>.50)
Representativen ess	9, 10, 11	0.251	0.222	not statistically significant (p; .445>.05)
Correlation implies causation	16	0.450	0.367	not statistically significant (p; .661>.05)
Equiprobability bias	13, 18, 19, 20	0.620	0.438	not statistically significant (p; .171> .50)
groups can only be compared if they are the same size	6	0.350	0.466	not statistically significant (p; .094> .50)

Table 2: The results for misconceptions subscales

CONCLUSION

The aim of this study was to compare the statistical reasoning types of gifted students and mainstream students. Among statistical measures, measures of central tendency are the most familiar topic for both types of Turkish students. The challenging part of these type of questions was to be aware of the outliers. The gifted students outperform statistically significant than mainstream students, thus it can be claimed that the gifted students are more careful about the outliers.

Another subscale which the gifted students did statistically well than their mainstream counterparts is *understanding independence.* Although the related topic is mainly taught in the 10th grade, the gifted students answered many questions as correct. That is might be explained with that the gifted students do mathematics like older students (Sowell, Zeigler, Bergwall & Cartwright, 1990). The scale of *correctly interpret probabilities* was measured by only one question. Although the gifted students were statistically better than the mainstream students, it is difficult to claim that the gifted students' performances were really good. Yet, the gifted students were good at following the multi-step question instructions. Additionally, the gifted students statistically performed better than their mainstream counterparts in the scale of *understand importance of large samples*. The related questions were chosen from daily life but requiring detailed analyses to make right judgements about appropriate sample size. This success might be explained with the fact that gifted students do well in daily life probability questions by using intuitive thinking skills (Baltaci, 2016).

Overall, although many of the questions are not covered in the Turkish national mathematics curriculum, the gifted students statistically outperformed at half of the subscales of correct reasoning types. This result shows that there is a relationship between the mathematical reasoning abilities of gifted students with their statistical reasoning abilities.

On the contrary, our study results show that both types of students' performances are very low in computing probability which is related to lack of combinatorial reasoning. Consistently, they have equiprobability bias strongly. This is because of most of the topics of this scale were the topics of Turkish 10th-grade mathematics curriculum. One may expect that gifted students perform beyond their grade level, but in statistical reasoning test gifted students did not perform significantly higher than mainstream students for many subscales. There is not much difference between gifted and mainstream students in terms of statistical reasoning when they both have limited knowledge of probability/statistics with limited experience in school. In order to develop correct reasoning skills, mathematics curriculum was required to focus on more statistical reasoning besides computing only some measures of statistics. Additionally, both groups have many misconceptions related to statistics but there were no statistically significant differences among the both groups in any misconception subscale.

Results obtained from this research yield some possible suggestions for further research. Since giftedness is a domain specific phenomenon (Renzulli, 2002), statistics course should be redesigned to introduce such thinking strategies to gifted students. Besides, there was no available information related to mathematical abilities of the gifted students prior to the study. The reason why the gifted students did not statistically outperform in all subscales than the mainstream students might be the fact that every gifted student in the study is not good at mathematics. Additionally, participants in this study were gifted and mainstream students who were studying in Istanbul. Since participants were chosen purposefully, not randomly, statistical reasoning of other gifted and mainstream students' needs to be examined in order to improve generalizability of the study. Therefore, more studies with different participants especially from other regions of Turkey, need to be conducted in order to have an idea about statistical reasoning of Turkish high school students.

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IMPROVING MATHEMATICAL MOTIVATION FROM MATHEMATICAL CREATIVITY WORKSHOPS

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Abstract. *This article presents an analysis about the improving mathematical motivation from mathematical creativity workshops. The research occurred with a group of fifty students of the last year of the Brazilian High School, in a public school of Brasília / DF - Brazil. The students were separated in two groups: control and experimental. The first one took a set of seven mathematical traditional classes (control), while the second one, participated of seven mathematical creativity workshops (experimental). A Mathematical Motivation Scale was used before and after this period. By results, it was verified a significant increase in the level of mathematical motivation of the experimental group.*

Key words: Mathematics Education. Mathematical Creativity. Creative and Critical Thinking. Mathematical Motivation.

INTRODUCTION

For some time, mathematics learning has been considered as a big challenge in basic education, especially in Brazil. Different national and international data show how distant is the quality mathematical education offered to the Brazilian students (IPM, 2016; Brazil/MEC, n.d; OECD, 2016; Saldaña, 2015). Therefore, mathematics learning is a topic that has raised concerns in different countries, specifically in the development of basic skills for 21st century citizens.

Part of the problems associated with school failure in mathematics is related to students' lack of motivation to learn this discipline. One of the factors that stimulate this problem is a non-contextualized teaching, mechanical and without opportunities of experimentation of the process of construction of mathematical knowledge. As a result, the discussion about mathematical motivation has been growing among different researchers, who consider this element fundamental for learning. It is worth mentioning that mathematical motivation is associated with mathematical creativity. (Fonseca, 2015, Gontijo, 2007, Havoold, 2016, Kattou, Kontonyianni, Pitta-Pantazi & Christou, 2013, Petrovici & Havârneanu, 2015).

According to Boruchovitch and Bzuneck (2001), motivation has sometimes been understood as a psychological factor, or set of factors, sometimes as a process. The motivation can be described by the interest, pleasure and satisfaction caused by the accomplishment of a task. It can also be perceived when the individual seek information, in their area of interest, to develop their domain skills.

Sternberg and Lubart (1999) consider that both types of motivation, intrinsic and extrinsic, are often in interaction, combining each other to strengthen creativity. However, they particularly highlight the influence of intrinsic motivation in the creative process, and how people are much more likely to respond creatively to a given task when they are moved by the pleasure of doing it. This aspect was observed in several studies reviewed by the authors with professionals who had been doing highly creative work in different areas and who were mobilized by the love of the task, focusing much more attention and energy on the work itself than on possible awards or recognition for its accomplishment.

Regarding the ability of mathematical creativity, the concept of Gontijo (2006) was adopted for this research, which considers it as:

> The ability to present numerous possibilities of solution appropriate to a problem situation (fluency) so that they focus on different aspects of the problem and / or different ways of solving (flexibility) it, especially unusual forms (originality), both in situations that require resolution and elaboration of problems as in situations that request the classification or organization of objects and / or mathematical elements in function of their properties and attributes, either textually, numerically, graphically or in the form of a sequence of actions (Gontijo, 2006, p.4).

It is worth mentioning that different authors mention that there are significant relationships between mathematical creativity and mathematical motivation (Fonseca, 2015, Gontijo, 2007, Grégoire, 2016 and Mann, 2005).

Alencar and Fleith (2003) still offer support for such findings, since they point to motivation as a component of creativity, still in the theory of general creativity - which is naturally expected to be continued in specific creativities. Fonseca (2015) emphasizes that motivation is a common element among different systemic theories of creativity, which induces to believe that it is not different in creativity in the field of mathematics.

In 2007, Gontijo conducted research with 100 Brazilian high school students whose aim was to investigate the relationships between creativity, creativity in mathematics and motivation in mathematics. On this occasion, the application of different instruments and statistical analysis found a positive correlation between mathematical motivation and mathematical creativity. Previously, Mann (2005) had pointed out that factors such as knowledge, personality, self-concept and motivation had an impact on creative development.

Thus, the purpose of this article is to present an analysis of the use of mathematical creativity workshops as a strategy to stimulate the mathematical motivation, based on a study conducted with fifty students from a public school of Brasilia / DF - Brazil. The hypothesis is that stimulate creativity also can helps stimulate motivation in mathematics.

METHOD

The students who volunteered for this research was separated in two groups. The first one with twenty-six students had a mean age of 16.85 years (SD = .9; Minimum = 15; Maximum = 18). The second one with twenty-four students had a mean age of 17.12 years (SD = .6; Minimum = 15; Maximum = 18). All volunteers were students of the last year of Brazilian high school - the last stage of basic Brazilian schooling in a low-class region of Brasília / DF - Brazil. The mathematics teachers of this school work to Education Secretary of Federal District for more than fifteen years working as teacher.

The Scale of Mathematical Motivation (Gontijo, 2007) is an instrument that consists of 28 items and its aim is to investigate students' level of motivation in mathematics. The items are evaluated in a 5-point Likert type scale, in which (1) is never, (2) rarely, (3) sometimes, (4) often and (5) always. Examples of items: Item 1 - Mathematics classes are among my favorite ones; Item 4- I am used to explaining nature phenomena using mathematical knowledge; Item 18 -

I Study Mathematics every day during the week; Item 16 - I am frustrated when I don't manage to solve a mathematical problem; Item 20 - I like to elaborate challenges involving mathematical notions for my friends and family; Item 36 – It's a pleasure to learn mathematics.

This instrument was applied at the beginning and at the end of the research for both groups. Between the first and the last application it took about a month and a half. During this period, the first group (control) took seven mathematical traditional classes, while the second group participated of seven mathematical classes that involved mathematical creativity workshops (experimental). For both, these classes were developed in the opposite time shift to the regular school activities.

To experimental classes, was used open-ended problems and creativity techniques. These problems served to enable students to engage in the task of creating diverse strategies, thus exercising creative thinking in mathematics and thereby positively connecting with it. At each meeting, students were able to hone their skills in terms of fluency, flexibility and originality of ideas. Of course, the mathematical content is also worked out of broader questions that admit, therefore, the possibility of the student hypothesizing and conjecture rather than just performing algorithmic application.

For example, while in a control group the plan was review numerical expressions using just board and pen, in an experimental group, students should talk in groups about what was necessary for this subject (using creative techniques such as brainstorming/ brainwritting). In these techniques, the participants should create a lot of ideas about a theme – this is helped students elaborate own review about the numerical expressions.

The result obtained from the scale was stratified into responses that demonstrate positive feelings and negative feelings about mathematics. The results were then analyzed using the student's t test to verify the occurrence of significant differences in motivation levels before and after of these mathematical classes for both groups.

RESULTS AND DISCUSSION

The Scale of Mathematical Motivation was applied in the first meeting with each group of students. In filling out this instrument, they indicated how often they identified each sentence expressed as behavior, act or attitude of their own during their academic, professional, and personal lives in relation to mathematics. Because it is a scale, there are no right and / or wrong answers, but the register of the subject's feeling about his relation to mathematics at that moment. After of this period, and more than a month later, the students were invited again to fill this scale.

A statistical analysis was then used to better understand the results found. And for this, the answers were stratified in sentences related to positive and negative feelings about mathematics in both groups, which allowed the elaboration of the following table (see table 1):

Group	Type of items	Application Test	Average	Variance	Stand. Desv.
Control	Positive values	1º Turn	70.42	544.28	23.32
		2º Turn	73.00	550.00	23.45
	Negative values	1º Turn	11.71	15.57	3.94
		2º Turn	11.42	14.95	3.87
Experimental	Positive values	1º Turn	68.14	403.47	20.08
		2º Turn	76.00	350.33	18.72
	Negative values	1º Turn	15.28	14.90	3.86
		2º Turn	14.71	16.57	4.07

Table 1: Results from Scale of Mathematical Motivation

As expected, factors linked to positive feelings in mathematics have increased, while factors linked to negative feelings in mathematics have decreased. And this occurred in both cases according table 1. It is worth noting that the variance and, consequently, the standard deviation, obtained a high decline when the experimental group was observed in relation to the positive values. This helps to understand that there was greater group cohesion in motivation in mathematics after the experiment.

This sounds an evidence that mathematical creativity workshops could help students to get involve more in math tasks, what was hoped according to literature (Fonseca, 2015, Gontijo, 2007, Havoold, 2016, Kattou, Kontonyianni, Pitta-Pantazi & Christou, 2013, Petrovici & Havârneanu, 2015).

According to verify if the occurrence of the change in values was in fact the existence of significant differences, we used the student's t test after checking the necessary assumptions. The following table summarizes the results (see table 2):

Group	Pair	T- Test	P- Value
Control	Positive values (1ª and 2ª Turns)	-1.36	.22
	Negative values (1ª and 2ª Turns)	.42	.69
Experimental	Positive values (1ª and 2ª Turns)	-3.21	.02
	Negative values (1ª and 2ª Turns)	2.82	.03

Table 2: Results of T-test

With the value $p < .05$, it can be inferred that there are significant differences between the two paired samples, which allows to conclude that the creativity workshops in mathematics may have contributed significantly to the increase of the expected motivation in mathematics and that, therefore, reinforces the initial hypothesis of this research. It worth mention that although the control group increased mathematical motivation levels too, the difference was not significant according to the test.

This shows that the workshops help increase mathematical motivation of the sample. And, for this reason, put methodologies like that could be useful to many students during the basic school. One way, although this is not the only, to help that students that initially don't like math.

To simulate if these results keep similar with a bigger sample, was used bootstrap option from SPSS. This way and considering a sample with n = 991, we found the same inference: $p<.05$ to experimental group in both cases (positive and negative values) and $p>.05$ to control group in both cases (positive and negative values).

CONSIDERATIONS

The stimulus to creative thinking in mathematics or just creativity in mathematics, seems to be an element that is related to motivation in mathematics, which, in turn, seems to have relations with proficiency in this area of knowledge. This is the main reason to believe that a work with the purpose of fostering these three variables together is essential today. It is important that students feel stimulated to think in mathematics, encouraged to do math.

Motivating students in mathematics can be the first step to engage in task of doing math on a daily basis, in academic and professional settings. Math-motivated students can devote more to learning in this area and thus become more proficient as well as creative in mathematics, matching what is expected of the citizens of the 21st century.

This study composes part of a broader research related to the doctorate, still in progress, of Fonseca[10], member of the group PI Research and Investigations in mathematical Education, at the University of Brasilia (UnB). The whole project aims to deepen research on relationships between what it calls the triad creativity-motivation-proficiency in mathematics - an area of study still filled with questions to be better understood.

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FOSTERING YOUNG CHILDREN'S CREATIVE MINDS: KINDERGARTEN KIDS EXPLORE SCHOOL-BASED STEM LAB

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Abstract. *The paper presents a study of kindergarten students building bridges using LEGO blocks in a school-based STEM lab. From the theoretical perspective, the study is grounded in the ideas of providing young children with challenging and complex tasks which would enhance fostering their creativity and ingenuity. The initial results make explicit how technology-rich environment stimulates students' interest in working on constructions using their imagination, creative potential, in collaboration with peers, while being perseverant facing obstacles and dealing with challenges.*

Key words: Makerspaces, STEM, complex problem-solving, design, creativity, ingenuity

CONTEXT OF THE STUDY

In June 2018, at one rural middle school of Southern-Eastern New Brunswick, Canada, visitors of provincial makerspaces school fair could see a poster at the entrance of Gala showcase. The poster presented portrayals of genius of Science, Technology, Engineering, and Mathematics of all times, and also … photos of some of the students whose talents have been celebrated during the event. This paper builds on the previous work on linking technology, challenge, enrichment and creativity in mathematics as connected to other STEM disciplines (Freiman, 2009 and Lirette-Pitre; Sriraman and Freiman, 2011; Blanchard et al, 2010), and particularly to the talent development in young children (Freiman, 2018).

Creativity was also a focus on two recent volumes by Springer: *Creativity and Technology in Mathematics Education* (Freiman and Tassell, eds., 2018) and *Computations and Computing Devices in Mathematics Education before the Advent of Electronic Calculators* (Volkov and Freiman, eds., 2018). The authors have contributed a chapter to each of these volume addressing the richness of Piagetian and Vygotskian theories in a context of mathematical problem-solving and creating music with iPads (de Champlain, et al., 2018) and the history of invention of mechanical calculators (Freiman and Robichaud, 2018), both could apparently be linked to students creative work in novel technology-rich environments now being introduced to local K-12 schools. More specifically, our study seeks to deepen our collective understanding of the phenomenon of makerspaces in terms of their potential to foster STEM-related talents and creativity in young children starting from early schooling.

Makerspaces are known as learning spaces (digital fabrication labs, or 'fabulous labs', introduced in early 2000s in the United States; Gershenfeld, 2012). They present different layouts, in which students engage in a multitude of projects, during which they explore various technologies, create new things of all kinds, and share their results with others (Brilliant Labs, 2017). Thus, these labs provide an environment in which students can design, experiment, build and invent while learning about STEM. Activities can range from cardboard construction to electronics, programming, robotics, and sewing. These informal activities became increasingly popular in 2000s with the advent of Information and Communication Technology (ICT).

According to Oliver (2016), makerspaces take many forms but generally involve a physical (often non-classroom, e.g., in public libraries) space with shared resources to pursue technical projects of personal interest with the support of a maker community. Thus, "making is more commonly practiced in after-school camps and clubs, making has the ability to enrich the school-day curriculum and bridge formal and informal learning contexts" (Oliver, 2016, p. 160). In a K-12 context, according to Niederhauser and Schrum (2016), there is a "relationship between the maker movement and the effort to increase STEM-related curriculum and interest in STEM careers and to move beyond current career" (p. 329). According to Peppler and Bender (2013), the latter helps students to "make their own jobs and industries" (as cited in Niederhauser and Schrum, 2016, p. 329).

Besides establishing and promoting learning in a variety of technology-rich environments, Hagel et al. (2014) emphasize the involvement of "different players acting in the maker ecosystem – beginners, collaborators, and market innovators" (as cited in Oliver, 2016, p. 160) in the maker movement. Referring to Maker Media (2013), Oliver (2016) also argues that makerspaces can be implemented across K-12 grade levels, to include easier electronic circuit projects and programming languages like Scratch for the elementary grades and more challenging 3D modeling and programming languages like Arduino for middle and high schools. In this (nearly informal) context (disregarding the concrete location of the makerspace), learning is seen as both an autonomous activity (self-directed learning, problem-solving) and a networking activity (self-determined learning, problem posing, and collaboration with others to build together a path of investigation) which also can foster new forms of technological creativity which helps them to incorporate mathematics in a way which goes beyond existing curriculum in line with so-called *soft-skills*.

Makerspaces are a relatively new phenomenon in the K-12 education system, which has gained ground during the past decade in many countries, including Canada (Sheridan et al., 2014; Hughes, 2017). In New Brunswick, first school makerspaces were established in 2014-2015, initiated and piloted by Brilliant Labs, a non-profit group that "supports integration of creativity, innovation, coding, and an entrepreneurial spirit within classrooms and educational curricula" (Brilliant Labs, 2017) across Atlantic Canada. While an exact place and role of makerspaces in the provincial schools is not yet clearly defined, their close connection to STEM disciplines is aligned with one of the objectives of the new 10-Year Provincial Education Plan (GNB, 2016) clearly oriented towards a competencies-based, career-oriented learning along with a focus on mathematics (numeracy), science, and technology.

Since 2016, a team of researchers from the Université de Moncton involved in the partnership development of network CompeTI.CA (ICT competencies in Atlantic Canada) has been following makerspaces movement in NB schools from a variety of angles: digital literacy skills development, STEM education, and soft-skills (Freiman et al., 2017; Djambong and al., 2018; Léger and Freiman, 2018). As contribution to the MCG conference, we focus on one case of fostering STEM-grounded creativity in kindergarten students attending a primary suburban K-5 school situated in the south of New-Brunswick.

THEORETICAL AND METHODOLOGICAL PERSPECTIVES

Elements of maker pedagogy can be retraced in history. Indeed, the works of Dewey (1916), Piaget (1956), Vygotsky (1978), Papert (1980), and Lave and Wenger (1991) have influenced the STEAM movement ('A' stays here for Arts being embedded into STEM, see, for example, Litts, 2015). In fact, Niederhauser and Schrum (2016) consider "making" as a pedagogical orientation with its main focus on "integrating creativity and imagination with design and encourages problem-finding in addition to problem-solving" (p. 359).

Several possible learning benefits of this pedagogy are evidenced in literature. Interdisciplinarity is often found to be one of the main advantages of school makerspaces that have a particular focus on STEAM skills (Litts, 2015). Sheridan, Halverson, Litts, Brahms and Jacobs-Priebe (2014) argue that the work in makerspaces fosters students' autonomy and collaboration. Among other elements LItts (2015) mentioned the development of critical thinking and argumentative skills, and the capacity of problem-solving.

Overall, the researchers seem to agree that when working on their projects, students explore different possibilities for their future career choices (Litts, 2015), while becoming active members of their learning community (Sheridan, Halverson, Litts, Brahms, & Jacobs-Priebe, 2014). Moreover, this experience seems to engage and motivate young learners because they do work according to their personal interests (Litts, 2015). Some researchers have also observed increased perseverance and self-esteem among the learners (Blikstein, 2013).

In the context of small children, use of blocks among other construction materials was advocated since early 19th century in the work of Pestalozzi and Froebel ('gifts of knowledge') for playing and learning geometric forms and their relationships to one another through the pattern building and recognition (Wolfgang et al, 2003). The author also refers to Piaget (1962) to acknowledge potential of LEGO blocks (as modern forms of constructional blocks) to develop spatial relationships in young learners allowing them to build imaginary structures representing real objects. In his turn, Resnick (2007) emphasized the use of construction blocks by kindergarten students so as to "designing, creating, experimenting, and exploring", as well as "imagination of new ideas and new projects". In this way, students' approach to learning can also be connected to Papert's constructionist vision in line with its use of "building", "constructing", and "knowledge representing" as central metaphor of "learning by doing" (Harel and Papert, 1990).

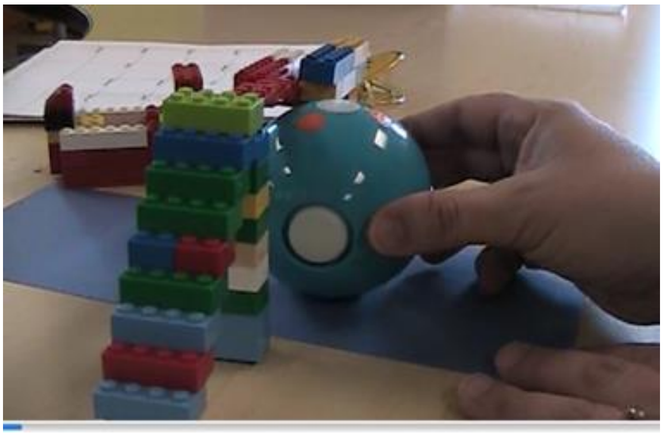
The process of creativity at young students is investigated when they are solving a complex task of *building a bridge solid enough to people and cars to safely cross it and high enough to let ships to pass underneath of the bridge*. The teacher working with a group of kindergartners aimed to unleash their imagination and, being equipped with a box of a variety of LEGO blocks, working with their peers (in groups of two or three) to design, to build, and to test their bridges. Each team was also given a blue band of construction paper to represent the river. After having obtained an ethical approval and parents' consent, with video-recording of students during their work, we were able to make observation of how their approach the task, what challenges they face, decisions they make in order to complete their projects. Below, we present some initial analysis of our data on this on-going study.

REFLECTING ON SOME EXAMPLES OF STUDENTS' INGENUITY WHEN BUILDING BRIDGES

Due to the limited spaces and preliminary character of the analysis, we discuss few examples from students' work which make explicit their creative problem-solving process.

Episode 1. Teacher tests student's construction: it does not let the ship to move under the bridge.

Two students have completed their task by building a bridge. They look happy with their result. The teacher comes and brings a 'ship' (in fact, a big ball representing the ship) and tries to push it underneath. Indeed, it did not fit. She asks the students: "Now, what do we need to do?". Students have now a new challenge to solve:

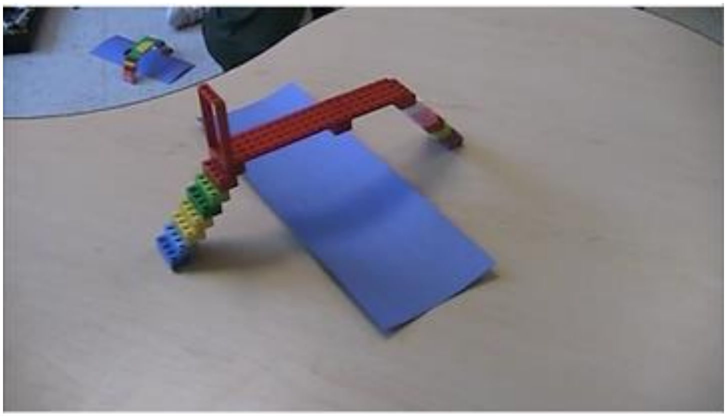


Episode 2. Students test it again after having made some adjustments.

The same team of students (Episode 1) has made the adjustments to their construction (Researcher: *What did you do? You added one block to each side?* Student: *No, we added two …*). Now, students had to test their construction … and their bridge has passed the test, the 'ship' could move under the bridge:

Episode 3. Students made initial construction by building 'stairs' on each side.

Another pair of students had a different design with symmetrically built stairs on both sides of the bridge they had yet to test:



Episode 4. Fixing a construction when it got broken.

When the students were testing their construction (*Episode 3*), their bridge got broken (not solid enough). Researcher: *What are you going to do?* Student (with confidence): *We are going to fix it.* Interestingly, the students, in order to fix their construction, started to re-build their bridge 'from scratch' taking blocks of different colors, keeping the same type of design:

Summary of initial observations: Overall, all students were clearly **enthusiastic** when approaching the tasks. While demonstrating a **variety of ideas** of how to design their bridges thus combining creativity and ingenuity when trying to select appropriate LEGO pieces, to use them to make their constructions, many of which represented **complex configuration** of different shapes for the main part of the bridge and approaches to it. Student had to **constantly monitor** different constrains (for example, to make sure their bridge is long enough to allow crossing the river). Some students were using **ideas of symmetry**.

During their process of building, students were facing a complex task and had to deal with a number of challenges, like ones shown in Episodes 1-4 (The ship does not fit to go under the bridge – the bridge is not high enough – Episodes 1-2; the bridge is not solid, there is a need to fix it). In these situations, despite the challenge and failures, students have demonstrated high level of engagement and perseverance when trying to accomplish the task.

The context of makerspaces and STEM-grounded **design tasks** seems to boost creativity and ingenuity which are the aspects often neglected by aspiring mathematicians, according to MacFarlane (2016). Conforming to the author, in today's complex world and its problems people face, these qualities provide them with adequate problem representation skills in the situations where the answers are not readily apparent. Moreover, our research findings, even in the preliminary stages, point into the need to educate students to handle the complexity of real problems early in their schooling. Yet more research is needed to get a more complete understanding of the learning culture which emerges from these early experiences, and in what way it might enrich further creative learning of all students.

CONCLUSION

In the context of makerspaces, kindergarten students participated in our study had an opportunity to explore different technologies (digital and non-digital ones) within complex and open-ended tasks (Freiman, 2018), which could spark their creativity, ingenuity, and inventiveness. While not being explicitly connected to mathematics curriculum, the task seems to provide students with authentic and complex problem situation in which mathematics has emerged being embedded and embodied into their ideas, actions, and interactions with their peers and technology-rich environment (STEM-lab) thus creating dynamic learning conditions which would enable, engage, encourage, enrich, and empower their 'transforming self' creativity (Freiman and Tassell, 2018; De Champlain et al., 2018).

Further, our results open to new practices and research paths which could provide deeper insight into how creative tasks, such as 'building with technology', would promote novel 'faces' of mathematics which is more collaborative, discovery-oriented, materialist (in terms of affordances of technology tools) and authentic (being part of open-ended didactic engineering) thus featuring richer connections between mathematics as an academic subject and mathematics as a creative endeavor which has its deep roots in human history, such as inventions of computational devices in 17-19th centuries (Freiman and Robichaud, 2018).

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CREATIVE AND CRITICAL THINKING IN MATHEMATICS: A WORKSHOP FOR TEACHERS

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Abstract. *Increasingly critical and creative thinking is advocated on the international stage as a necessary capacity for 21st century education. In Brazil, this subject is still something new, especially for the primary education teachers. In order to contribute to the stimulation of critical and creative thinking in mathematics of students between 6 and 10 years old, was organized a workshop for teachers of these school years. The aim of this study was to proceed with a brief study of the validation of this workshop with a group of 27 teachers of primary education. As a result, the unanimity of positive feedback about the workshop and the reports collected from the sample show that it has acceptance and potential for continuing teacher training.*

Key words: Mathematics Education. Mathematical Creativity. Creative and Critical Thinking in Mathematics.

INTRODUCTION

Several researches have been conducted with the focus on developing students' creative skills. However, few studies are found related to courses and teacher training programs aimed at developing the creativity of those who conduct training activities in educational institutions. In a study of 7,659 teachers from different levels of schooling in 32 European countries, Cachia and Ferrari (2010) pointed out that there is a discrepancy between how teachers perceive creativity and how they claim to stimulate creativity during teaching. Teachers' opinions about creativity in education are stronger than their practices. According to the authors, this implies that there is space for improvement in the way creativity is fostered in schools. Regarding creativity in mathematics specifically, Leikin (2009, 2010, 2011) points out that studies that involve the conceptions of teachers are underdeveloped in mathematics education research. Aktaş (2015) suggests that standardized tests, curricular limitations and the educational system act as some of the barriers to promoting creativity in the educational setting.

In the Brazilian scenario, with regard to creativity in mathematics, few works were developed. According to data collected from the Brazilian Digital Library of Theses and Dissertations, of the Brazilian Institute of Information in Science and Technology, in the last ten years, between theses and dissertations, only 17 productions were defended in the country. The results were obtained using the descriptor "creativity in mathematics" in the "Title" and "Subject" fields. In the first case, 16 papers were obtained, with a repetition of 3, resulting in 13 papers. In the second case, 21 papers were obtained, with a repetition of 3, resulting in 18 papers (of which, two are unrelated to mathematics - "Developing Criticality and Creativity with Geography Students through Modeling" and "The Creativity Map: Validity of construct through the analysis of networks "), resulting in 15 works. Comparing the results of the search with the descriptor "creativity in mathematics", considering the fields "Title" and "Subject", we have 20 works, being 17 works in the last ten years. Of the 17 papers found, 3 are theses and 14 dissertations. These data also reveal that 3 studies focused on training processes and/or teaching performance; 2 investigated the early years of elementary school; 4 studies had the final years of elementary school as object of analysis and 8 related to High School. These data highlight the small number of researches on creativity in mathematics in Brazil, developed in Postgraduate Programs.

Among the studies that investigated the processes of training and/or teaching performance in Brazil, we highlight the research developed by Silva (2016), who after a long period of observation of teaching practices in the classroom, found evidence of stimuli that can influence the development of students' mathematical creativity, however, these stimuli are realized intuitively by the teacher, without a systematic planning of activities aimed at the development of students' creativity.

Another work to be highlighted was produced by Farias (2015), who found that the students' perception about the school evaluation, as well as the teaching methodology used by the teacher has contributed significantly to the development of creativity in mathematics, however, in a negative way. Methodologies supported by a transmissive conception of teaching, such as "only concerned with information content", "always uses the same teaching methodology", "offers students little choice in the work to be done", do not favor the development of creativity in the field of mathematics. Soh (2017), also reinforces that the way teachers act plays a fundamental role in promoting students' creativity.

Programs and/or teacher training courses can be a way for changes in pedagogical practices. In these formative spaces, it is possible to stimulate some attitudes of teachers that promote creativity: (a) to encourage students to learn independently; (b) have a cooperative and socially integrative teaching style; (c) motivate their students to master factual knowledge so that they have a solid basis for divergent thinking; (d) not judge students' ideas until they have been carefully worked out and clearly formulated; (e) encourage flexible thinking; (f) promote student self-assessment; (g) provide opportunities for students to work with a wide variety of materials and under different conditions; (h) help students learn to deal with frustration and failure so that they have the courage to experience the new and the unusual (CROPLEY, 1997).

From this lack of stimulus to creativity in mathematics with students and teachers, a project has been developed with a public school in an administrative region of the Federal District. The focus includes motivating the characters of the school institution (students and teachers) to exercise their creativity. This research includes application of tests, scales and inventories before and after intervention along with a series of 10 meetings with students.

As for the group of teachers, were developed 5 workshops, such as (a) patterns in calendars, (b) balloon cars, (c) healthy consumption, (d) geometric shapes (tangram) and the giant shoes. Although there were no tests and scales applied to the group of teachers, were collected demographic data from the teachers, as well as an evaluation form was completed at each end of activity.

In this way, this work aims to report the workshop titled giant shoes. The purpose of this article is to proceed with the validation process of this workshop for teachers with a view to stimulating critical and creative thinking in mathematics.

For the development of the workshop, critical thinking was considered as a multidimensional construct, which implies both deductive and inductive reasoning processes to achieve a desired result (Wechsler et al., 2018). And, with respect to creativity in Mathematics, it was considered

> The capacity of presenting several solutions that are appropriate for a problem (fluency) so that these focus on different aspects of the problem and/or different forms of solving it (flexibility), especially unusual ways (originality). This ability can be employed both in situations that require a resolution and an elaboration of the problem, or in situations that demand a classification or organization of objects and/or mathematical elements according to their properties and attributes, in textual, numerical or graphic format or in a sequence of actions (Gontijo, 2006, apud Gontijo; Fleith, 2014, p. 68).

METHODS

The training workshop to stimulate critical and creative thinking in student mathematics was carried out with 27 teachers from a public school in an administrative region of the Federal District - Brazil (22 females and 5 males) working with children in the age group of 6 to 10 years old. All participants had undergraduate courses, with a predominance of training in the Pedagogy course. Only 2 teachers stated that they had some reading and/or participation in courses designed to discuss the theme of creativity in the school environment.

In this text, an activity will be described whose main theme was work with proportions, content that is foreseen in the National Curricular Common Base - BNCC (BRAZIL, 2017), which guides the formulation of curricula throughout Brazil. It should be noted, according to the BNCC, that proportionality should be, for example, present in the study of: "operations with natural numbers; fractional representation of rational numbers; areas; functions; probability etc. Moreover, this notion is also evident in many everyday actions and other areas of knowledge, such as sales and mercantile exchanges, chemical balances, graphic representations, etc. "(p. 266). This ability is present in the school curriculum from the 4th year of elementary school.

The workshop had three main moments: a) Warm-up and approach with the task in the proposition of enigmas whose objective was to initiate a work of divergent thinking; b) Presentation and resolution of the problem: "Suppose you are giant and that this shoe nominated by Guinness World Records (2012) is yours, how would you do to know your height?"; and, c) Construction of the concept/definition, where it was proposed to systematize the results built during the workshop for the problem. In addition, in this last phase there was also space for discussion of the solutions found and discussion on how to conduct activities such as this among students from 6 to 10 years old of primary education.

In the midst of these phases, the participants had moments of written production:

1) Individual work, with registration of the hypotheses of solution on a sheet of paper (it is emphasized that it was requested to elaborate hypotheses, ways of solution and not the solution itself).

2º) Individual work, to analyze the hypotheses formulated by a colleague. In this activity, teachers were instructed to record their opinions on the hypotheses presented in the colleague's sheet, making it through positive comments, but also expressing their doubts and opinions. Some expressions have been suggested for this record, such as: I agree with ... because ...; I disagree with ... because ...; I would like to add ...; I noticed that...; You could give another example of ...; So what are you saying is ...?; Do you think that...?; Could this not be ...? Why do you think that?; Could you explain what you said ...? Could you talk a little bit more about this ...?

3º) Return of the productions to their respective authors. Each participant analyzes the opinions they have received and, if so, re-register their ideas, incorporating the suggestions received.

4º) Dissemination of the hypotheses and/or solution paths from the composition of a mural.

5º) All the participants go to the mural to read the productions of their colleagues, classifying them as: "good", "better", "excellent", from colored labels (one color for each type of classification). This type of classification, according to Dacey and Conklin (2013), aims to stimulate the students' production, evidencing that all are able to propose mathematical ideas, although some of them may be more appropriate than the others.

6) Selection of the hypotheses and/or strategies considered more appropriate for the solution of the problem, without the interference of the teacher. After the selection, the teacher enters the scene problematizing the choice and offering the necessary subsidies to solve the problem, leading to the systematization of the concepts/definitions involved in the activity.

RESULTS

At the end of the meeting with the teachers, a form was distributed to evaluate the creativity workshop, through which the participants expressed their opinions about the activities developed. For each item evaluated, respondents could choose one of the five response options: 1 - optimal; 2 - good; 3 - regular; 4 - bad; 5 - lousy. Regarding the mathematical topic addressed, 83.3% reported that it was approached optimally and for 16.7% this approach was good. Regarding the clarity of the presentation and the quality of the activities performed, 100% of respondents indicated that these were optimal.

The evaluations stopped presenting unanimity when the evaluated question was the applicability of the workshop in the teaching practice with the students, nevertheless, the answers were still positive. For 58.3% of the respondents, "application" was considered "optimal", while for 41.7% it was considered "good". The rationale for "good" classification lies in the lack of confidence in the replication / adaptation of the workshop, in the amount of time needed to carry it out, and in the lack of an appropriate environment to develop differentiated activities. These explanations are consistent with the aspects pointed out by Aktaş (2015) that act as barriers to promote creativity in the educational setting.

According to the good evaluation of the workshop, it is clear that the group accepts this type of training activity. In addition, the collected testimonies allow to understand the immersion that the teachers were in that moment about the critical and creative thought in mathematics. Knowing an activity that stimulates what is called critical and creative thinking was new to this group of teachers, who on the occasion became students. The unfolding of this, and interesting finding of this validation process, is that they seem to have been provoked to bring this type of activity to their classes.

CONSIDERATIONS

The validation of this workshop from a small group of teachers signals some contributions. The first one refers to the fact that such a workshop can be replicated to other teachers so that they have a first contact with the subject matter in question.

A second contribution concerns the need to create a teacher training program with a view to presenting the concept of critical and creative thinking and allowing such professionals to experience strategies to stimulate this capacity. After all, in order to stimulate the creativity in mathematics of the new students, it is necessary to stimulate the creativity in mathematics with the teachers.

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“BUT PAINTING IS MORE FUN” ON INTERVENTION POSSIBILITIES FOR UNDERACHIEVERS IN MATHEMATICS EDUCATION

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Abstract. *Among the group of gifted children there is also the group of underachievers. Underachievement is conceptualized as a considerable discrepancy between high potential and poor school attainment. Underachievement is quite heterogeneous in its appearance and therefore requires individual support strategies in the classroom. In this article, concrete manifestations of underachievement and possible intervention strategies in regular mathematics teaching are presented based on extracts from a case study. This presented case study is part of a larger research project on underachievement in a ninth-grade mathematics class.*

Key words: giftedness, expertise, underachievement, interventions in mathematics lesson

INTRODUCTION

“But painting is more fun.”, that is Clemens sentence in a mathematics lesson. Clemens is a 14- year-old student in a ninth-grade of a grammar school in a small town near Leipzig. His sheets in the mathematics booklet are often bordered with drawings. He wishes that only Sudokus are made in mathematics lessons. John, a 16-year-old boy, also learns in this class. He repeats the ninth grade and does not write anything down in the mathematics class and does no homework. Lukas and Carl also attend this class, two 15-year-old boys who hardly take part in mathematics lessons and disturb the lessons with secondary activities. These are four students in a class the head teacher describes as gifted. At present, these four students do not show their high potential in mathematics class. These are pupils about whom Brandl (2011) says that not all mathematically gifted students have high achievement in mathematics.

If one asks Clemens what has happened since his elementary school days, when he skipped a school year due to his outstanding performance, he says: "Everything has disappeared.” These words show that these students need support.

These four pupils are examples of underachievers and show different forms of underachievement in mathematics teaching. Underachievement is a form of unfavourable giftedness development alongside other forms such as discrepancy in giftedness profiles (Nolte, 2015) and twice exceptional children (Nolte, 2012).

There are a lot of research findings on the causes and appearance of underachievement in psychological and educational research. However, Preckel (2011) notes that there is a research deficit with regard to intervention options in the case of underachievement. So far, underachievement has seldom been addressed in mathematics didactic research (e.g. Mellroth, 2013; Suan, 2014).

Therefore, the aim of this article is to show which intervention possibilities there are concretely in mathematics teaching with regard to underachievers based on exemplary excerpts of mathematics teaching in the class of Clemens, John, Lukas and Carl. The focus will be on the appearance and the intervention at Clemens.

First, the theoretical framework concerning mathematical giftedness and underachievement is given. This is followed by insights into empirical research with regard to methodology, results and discussion to underachievement in mathematics teaching.

THEORETICAL FRAMEWORK

Underachievement can arise on all performance levels. Underachievement is best researched in relation to giftedness. According to a categorical conceptualization, underachievement means a clear discrepancy between a high cognitive potential and the school performance shown (Rost, 2007). Underachievement is understood as a problem system based on a complex multifactorial causal structure of personal, family and school-related factors (Johansson, 2013; Preckel, 2011). Underachievers can be characterized by problems in their motivation to learn and to achieve, in their engagement, in insufficient self-control and in a negative self-concept. They need support to set up their learning- and work organization, to strengthen their self-control as well as adequate challenges in the lessons (Johansson, 2013). Ziegler et al. (2008) point out another cause. Their empirical study showed that fine motor deficits can determine underachievement. But there is no concrete intervention strategy regarding underachievers for teaching.

There is an empirical study that investigates an area-specific underachievement in mathematics (Suan, 2014). In his study, Suan examined 56 graduates of a Philippine teacher training college who were previously identified as underachievers exclusively in mathematics. Suan used a questionnaire with a five-point Likert scale divided into three categories: teacher factors, student factors and environmental factors. For data evaluation the author used the Chi-square test on relationship between mathematics performance and teacher factors, between mathematics performance and student factors and mathematics performance and environmental factors. Suan only found a significant correlation between mathematics performance and student factors. He concluded that student factors such as study habits, attitudes and interests toward mathematics and time management directly affect the attaining in mathematics.

The significance of this study is difficult to evaluate, because only a summary of the study is available. The study examines students in tertiary education. According to the author, the selected students show a partial underachievement only in mathematics.

The phenomenon of underachievement shows that giftedness does not have to be manifested in adequate performance. In gifted research, giftedness is regarded as a dynamic construct. There are a number of features that characterize a mathematical giftedness. In addition, there are personality traits such as intellectual curiosity, effort and persistence. In order for these skills to develop and lead to outstanding achievements, appropriate learning processes and support are necessary (Käpnick, 1998; Fuchs, 2006).

Fritzlar (2015) has linked this dynamic understanding of giftedness with the mentioned aspects with issues of the expertise approach. As a result of this integration he constructed the model of developing mathematical expertise. Innate structures in the brain are largely the starting point for the development of mathematical expertise. The key element for the further development of expertise is mathematical activity. Motivation and interest are also important. This model by Fritzlar illustrates that in the interaction of the cognitive apparatus, the intrapersonal factors, the environment and mathematical activity, mathematical skills and knowledge can develop.

Mathematical skills are the ability to memorize mathematical issues by drawing on identified structures, the ability to search for, construct and use structures, the ability to transfer and generalize structures, the ability to switch between modes of representation, the ability to reverse lines of thought, the ability to capture complex structures and work with them, mathematical sensitivity and mathematical creativity. Mathematical knowledge and skills can be displayed in special mathematical achievements. In his model, Fritzlar indicates this only as a possible option by dashed arrows. By this marking he refers explicitly to the possibility of underachievement. The model makes it clear that it is only a possibility and not a necessity that mathematical knowledge and skills lead to special achievements.

EMPIRICAL RESEARCH

Aim, research question and method

The aim of this empirical study is to show how underachievement in a ninth-grade mathematics class is presented in very different ways and how underachievers can be promoted in regular classes. This study aims to answer the question of how teachers can promote underachievers in regular mathematics lessons so that underachievement can be alleviated.

Individual case studies are used by empirical design. For this study, the following available data records are taken into account:

- protocols of participating observations in mathematics lessons,
- available school reports from primary school and grammar school,
- students' own productions from lessons and tests,
- interview protocols with teachers and parents.

In the following, the appearance of underachievement and the teaching intervention relating to Clemens will be analyzed as examples.

Empirical results: excerpts from the individual case study by Clemens

The school file at Clemens grammar school contains a document from the primary school, the educational recommendation (March 2014), which recommends the type of school Clemens can attend from the fifth grade. The text of this educational recommendation is quoted below (translation by the author).

> "Clemens is a quiet, attentive student. He follows the teacher's explanations with interest and criticism. He quickly grasps connections and recognizes essential things. He thinks analytically, identifies problems and develops solutions independently. He is happy to discuss lesson content with the teacher. Because of his ability to think and analyse logically, Clemens has skipped the first grade. In the written area he often needs a lot of time to solve exercises completely."
>
> school grades in the education recommendation:
> german 2[11], mathematics 2, science instruction 2

[11] In Germany there are six school grades: 1 very good, 2 good, 3 satisfactory, 4 sufficient, 5 deficient, 6 insufficient.

The educational recommendation shows good school grades so that Clemens can attend the grammar school. In general, Clemens is certified as having very good cognitive perception and interest in subject. This is also given as a reason for skipping a class. The only problem mentioned is Clemens slow speed in writing. In the school file there is not any documentation about Clemens problems in writing. For Clemens, there was also no appropriate support for this handicap. Because it is still present, it could be fine motor problems.

Furthermore, in the school file the grades of all school years are listed, which in the following will only be given for mathematics.

date of certificate	grade in mathematics
first half year class 5	3
end of year class 5	3
first half year class 6	3
end of year class 6	3
first half year class 7	5
end of year class 7	4
first half year class 8	4
end of year class 8	4

Table 1: school grades in mathematics by Clemens

The table shows that Clemens no longer received good marks in mathematics at the beginning of his school years in grammar school. In the first half of class 7, he even received a grade of 5 and is therefore considered to be at risk of failing the class. Towards the end of eighth grade this occurred again. He got the grade 5 again and should have repeated the school year. The school gave him the chance to take an assessment test. He received a grade of 4 in this test, which allowed him to be transferred to grade 9. In general, the school years at the grammar school show that Clemens could use his potential less and less and that his school career at the grammar school is in danger.

The course of school grades alone shows in this example how a gifted pupil who showed exceptional achievements in primary school has become an underachiever over several school years.

At the beginning of grade 9 (August 2018) there was a risk of further decline. Clemens did not take part in the mathematics lessons completely. Therefore, the aim was to support Clemens in mathematics so that he could reach the class goal of the ninth grade. For the author it was important that this promotion should take place in regular mathematics lessons to be able to show that regular teaching offers sufficient possibilities for fostering underachievers. In October 2018, the author decided to record Clemens' behavior in class and use concrete individual support measures in accordance with the general intervention strategies for underachievement (e.g. Preckel, 2011).

The evaluation of 32 protocols from mathematics lessons in the period from 10/22/2018 to 12/17/2018 shows the following problems: Clemens is often involved with painting in class, playing with pens and talking to his bank neighbor[12]. He then starts writing notes or exercises later. He writes very slowly so that he misses lessons or cannot finish exercises. More often he goes to the toilet during lessons, which takes longer. Clemens often has no homework. Clemens own productions show large gaps in mathematical knowledge. Clemens always approaches problem solving tasks with motivation, but does not find a solution.

As intervention strategies, we use non-verbal and verbal remembering of the use of learning time, systematic control of written work such as transcripts and exercises, consistent feedback during and after the lesson on its learning outcomes and behavior in the classroom. During practice phases, the teacher tries to work together with him on tasks by giving him scaffolding through mundane hints. In addition to this promotion of self-regulatory skills, Clemens also needs mathematics related support. According to Fritzlar's model, Clemens first gets help to close the gaps in mathematical knowledge. Furthermore, Clemens is promoted in finding his way in complex mathematical situations and in using mathematical structures (Fritzlar, 2015).

Some results of this intervention are outlined in the following. On 11/30/2018 Clemens took part orally in the lessons for the first time. He reported himself at the oral repetition and he correctly named a power law.

A short protocol excerpt of 12/3/2018 is given below.

> During the break Clemens comes smiling to the teacher and says: "I have my homework today." The teacher lets Clemens present the whole homework at the control. All tasks are solved correctly. He gets a smiley[13] for it.

On this day Clemens did his first homework in mathematics class in ninth grade. He communicates this joyfully excited already in the break and would like to perform the homework also. The teacher fulfils this wish and rewards this effort with a smiley so that Clemens can learn that performance can have positive consequences.

The following protocol section of 10/22/2018 shows both the mathematics-related problems and the deficits in self-regulation.

> At the beginning of the mathematics lesson, the teacher reiterates the power laws with negative exponents. For each power law, examples are repeatedly demonstrated on the blackboard. Afterwards the pupils ought to solve 10 similar tasks without help, for example: 2^{2^2}, $\frac{2^3}{5}$, $(3^2)^2$, $(\frac{4}{3^2})^2$. Clemens rummages in his bag. After four minutes the teacher tells him to start. He slowly writes down all the tasks without solving them. Then he paints again.
>
> Teacher: What do you need to solve the tasks?
>
> Clemens: I don't know how to solve them.
>
> The teacher explains the respective structure of the task with Clemens. Then she supports Clemens with structuring hints during the solution. So Clemens can solve two tasks.
>
> Teacher: Now you can solve all further tasks.
>
> She goes to other student tables.

[12] The observation will be continued. The data will then be evaluated quantitatively.
[13] For a certain number of smileys pupils can deserve a grade of 1.

Clemens talks to his neighbor. [The tasks remain unfinished.]

This section of the protocol shows the deficits in self-regulation with regard to the delayed start of work. Obviously, Clemens thinks he can't solve the tasks, he does not seek help, but begins to paint. Clemens does not answer the teacher's question. He generally says that he is not able to solve it. The teacher works with Clemens on the complex structure of the task so that he can apply the power law correctly. When the teacher leaves Clemens' table, he stops working. This sequence clearly shows that Clemens is not able to apply power laws to a task with a more complex structure. According to Fritzlar's model, Clemens needs support to see mathematical structures and to apply mathematical laws to them. This can only be achieved by mathematical activity with individual hints.

The following extract of the protocol (11/2/2018) gives an insight into Clemens' attempts to solve a problem. The following problem solving task should be done*: A passenger train departs from a railway station at 7:05 a.m. at an average speed of 40 km/h. At 7:35 a.m. it is followed by a fast train with an average speed of 70 km/h. Calculate when the fast train will overtake the passenger train* (TU Darmstadt, 2018).

Clemens starts immediately with the task and works the whole time on it. According to his written aspects it is obvious that he tries longer to solve it with the rule of three. He comes to no result.

Clemens: One has to do it with a diagram.

Teacher: Very good idea, you can do that. Or you try it systematically. Perhaps with a table.

Clemens: This will take longer.

This sequence shows that Clemens effectively uses learning time in this task. He does not realize that the rule of three can only be partially used here. He correctly recognizes that the use of a diagram could help here. This perception is supported by the teacher. In addition, she directs Clemens to the simpler strategy of systematic experimentation with a table. Clemens rejects this proposal because it is too time-consuming for him. Clemens names a correct strategy: the diagram. However, he is unable to apply this strategy. This can maybe be caused by concrete knowledge gaps and lack of experience with problem solving tasks. Obviously, Clemens fails here because of domain-specific knowledge and skills.

Since the end of November, the protocols show that Clemens is making better use his learning time in lessons. He does his homework more often. Since spring 2019 Clemens shows more and more that he is able to solve algorithmic problems well and that he is able to solve modelling tasks much more correctly. If at the beginning of the school year he received the grade 5 several times, he now achieves more and more the grade 3.

SHORT DISCUSSION AND OUTLOOK

The excerpt from Clemens case study shows that an existing potential was used in primary school and less and less at the beginning of secondary school. In this concrete case there is an extreme range from an acceleration of the school career in primary school to the danger of failing the class from grade 7. Clemens himself says it is submerged. The research situation on underachievement says that the greatest drops in performance occur in grades 5 to 7 (Johansson, 2013). This is the case with Clemens. If the teachers are asked, they agree that Clemens is still very infantile and that he simply does not learn. This would be congruent with the finding that factors on the student side are also responsible for underachievement (e.g. Preckel, 2011; Suan, 2014).

Another question that must be asked here is why there was no early intervention in the existing writing problems that were apparent in primary school. Although it is known in research on underachievement that fine motor problems can cause underperformance, the school has not promoted Clemens adequately.

The case study of Clemens also shows that underachievement can also be caused by school. Because four students in this class can be identified as underachievers, the question of the previous mathematics lesson must also be asked. A look at the exercise books of grades 5 to 7 shows that the teacher used only algorithmic tasks, dressed text tasks and only about seven closed problem tasks per school year. In such lessons, there were hardly any academic challenges that represented suitable opportunities for mathematically gifted children.

According to the model of Fritzlar, the underachievers of this class show that it is necessary that there are many possibilities in mathematics lessons to build mathematical experiences. It must be possible to use mathematical abilities. Mathematical expertise can only be built up if students are able to carry out large mathematical activities. The model of developing mathematical expertise by Fritzlar implies the promotion of mathematically gifted children for their further development. This includes investigative mathematical activity, building mathematical knowledge, opportunities to gain heuristic experience, and dealing with mathematical problems.

This case study shows that it is possible to give individual support to underachievers in regular mathematics lessons. For Clemens, it will be a long way back. Clemens is to be wished that he can find his potential again and find more teachers who can support him professionally.

This study in this ninth grade also points out that underachievement is always an individual phenomenon. Certainly, there are aspects that can be seen in several case studies on underachievers. Research on underachievement shows, that each underachiever has his own history and therefore needs specific interventions in the subject and in promoting self-regulation.

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"OH, I DO NOT LIKE THAT WHEN YOU HAVE TO JUSTIFY SOMETHING" – DIFFICULTIES IN FORMULATING ARGUMENTS AS A BASIS FOR THE SUPPORT OF MATHEMATICAL GIFTEDNESS

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***Abstract**: Arguments and argumentations are of high relevance in mathematics education, on the one hand as a learning goal and basis for proofing, on the other hand as a necessary basis for the deeper understanding of number characteristics and relations. The paper focuses on a qualitative analysis of the difficulties in formulating argumentation products of mathematically gifted primary students (age 9-11). With special regard to a structural analysis by means of the Toulmin layout, different categories of argumentation difficulties are specified. The difficulties are of contentual, methodological and motivational nature. After pointing out the difficulties, the results are taken into account for the support of mathematical giftedness with special regard to the support of argumentative abilities in order to exploit the full potential of excellent argumentative abilities.*

Key words: giftedness, argument, Toulmin, primary school

INTRODUCTION

The formulation of arguments is a central learning goal in and an important basis for mathematics education. Especially the view on mathematics as a deductive organized system with theorems and proofs forces fundamental skills according to argumentations (e.g. Hanna, 2000). Being a fundamental tool for proofing skills, it is already relevant in primary school and in the support of gifted students. Nevertheless, studies show that especially primary students have difficulties in formulating arguments in written and oral form (Cramer, 2011). Hereby, arguing has to be seen as a complex process in which difficulties can only be comprehended through the interaction of different aspects (e.g. Brunner, 2014; Nagel & Reiss, 2014). The idea that gifted students have less problems and difficulties during the formulation of mathematical arguments seems legitimate, even though empirical studies cannot verify this hypothesis clearly (Fritzlar, 2011; Jablonski & Ludwig, 2019). In the following, different influencing factors and difficulties in the formulation of arguments will be pointed out in the context of mathematical giftedness and its support.

THEORETICAL BACKGROUND

Arguments and Argumentative Competences

In the interpersonal communication, arguments are used to get the communication partner's consent through convincing statements (Cramer, 2011). Nevertheless, in mathematic lessons, arguments are mainly formulated in the sense of a deeper understanding of mathematics contents and relations (Baker, 2003). Arguments in a mathematical sense are strongly linked to related processes, such as justification, reasoning and proof. The paper defines "argument" as a product that derives from activities such as the formulation and questioning of assumptions on mathematical characteristics and relations (Bezold, 2009).

It is not separated from justification and reasoning, but distinguished from proof through deductive exactness.

Based on the term "competence" according to Weinert, arguing competences are skills that enable someone to understand arguments in verbal and written conversation, to produce them and to react to other arguments. Furthermore, they include the associated motivational, volitional and social readiness to successfully and responsibly use these reasoning skills in variable situations (Budke & Meyer, 2015). Difficulties in arguing or training in reasoning skills can therefore be of substantive or methodological nature. In addition to content knowledge, interdisciplinary – both linguistic and motivational – competences play an important role (Bezold, 2009). To enable statements on argumentative competences and difficulties, the analysis of argumentation products is necessary.

According to Toulmin (2003), every argument can be structured in its core with help of the layout that Figure 1 shows. It includes different functional elements, whereby the element "Data" is defined as "facts we appeal to as a foundation for the claim" and the "Conclusion" as claim "whose merits we are seeking to establish" (Toulmin, 2003, p. 90). The third element is called "Warrant" and stands for "general, hypothetical statements, which can act as bridges, and authorize the sort of step to which our particular argument commits us" (Toulmin, 2003, p. 91). In its original layout, the scheme involves further elements, such as the "Backing" as a support for the warrant, the "Rebuttal" for marking exceptions, and the "Modal Qualifier" for relativization of the argument's strength (e.g. Metaxas, Potari & Zachariades, 2009).

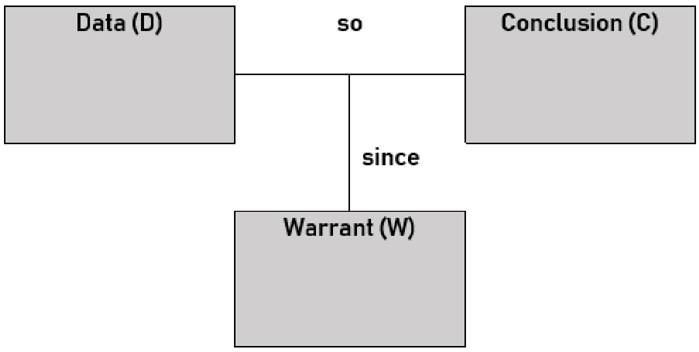

Figure 1: Toulmin Layout (see Toulmin, 2003, p. 90ff.)

With special regard to argumentative competences, it is necessary to analyze arguments according to their content. The structural analysis by means of the Toulmin scheme does not state anything on the argument's quality, e.g. whether a warrant is based on authority or a mathematical rule (Koleza, Metaxas & Poli, 2017). "An argument from D to C will be called analytic if and only if the backing for the warrant authorizing it includes, explicitly or implicitly, the information conveyed in the conclusion itself." (Toulmin, 2003, p. 116). Arguments that do not fulfil this condition are claimed as incomplete. These arguments leave open or fail to answer (further critical) questions (Koleza, Metaxas & Poli, 2017).

Argumentative Competences and Mathematical Giftedness

As the following study focuses on mathematically gifted primary students and their argumentative competences respectively difficulties and support, it is necessary to define the construct mathematical giftedness for this specific age and area. Further, the relation of argumentation and mathematical giftedness will be emphasized.

In the context of the study, mathematical giftedness is defined as a potential that might develop into an outstanding mathematical performance by means of an advantageous interplay of different influencing genetic, environmental and supportive factors (Käpnick, 1998). Diagnostic considerations are made by means of lists of specific characteristics of gifted students, such as the storage and structuring of mathematical contents, as well as personal characteristics, such as interest and motivation (Käpnick, 1998).

Previous research suggests that mathematical giftedness in elementary age does not solely affect argumentation (competences), especially not the need for argumentation (Fritzlar, 2011; Jablonski & Ludwig, 2019). Deficits in argumentation therefore affect all talent profiles, even though giftedness seems to influence important basics (creative detections and general findings) for outstanding argumentative competences (Jablonski & Ludwig, 2019). In the following, the difficulties of gifted primary school children are to be concretized and considered on a qualitative level as a basis for the support of mathematical giftedness.

EMPIRICAL STUDY

Within the enrichment program "Junge Mathe-Adler Frankfurt" [Young Math Eagles Frankfurt] for mathematically interested and gifted children of the primary and lower secondary level, a qualitative study on arguments and argumentative competences is carried out. The program has been realized in Frankfurt since February 2017 and is currently implemented with three groups of grades 3 to 5 (ages between 9 and 11 years). The children are nominated by their teachers by means of indicating tasks testing specific characteristics of mathematical giftedness (Käpnick, 1998). Every second week, the children can develop and deepen mathematical topics in the sessions with other children and the support of university students. Excursions into various mathematically related fields, the use of mobile technologies and participation in competitions emphasize the diversity of mathematics as well as its relevance to neighboring disciplines. The program focuses on mathematical content and competences, nevertheless argumentative competences are not trained to a higher extent than other competences.

The study is limited to verbal arguments, which is collected through task-based guideline interviews. The focus is on numerical lattices and number pyramids and special number relations in the formats, e.g. how the basic elements have to be arranged in order to maximize the result (see Figure 2). In May and June 2018, the data collection was carried out with 32 participating children (at the time of the survey in grades 3 and 4). The interviews were then transcribed, structured with the Toulmin scheme, and coded using Mayring's qualitative content analysis (Mayring, 2015). This serves as the basis for the following analysis, with the aim of identifying children's difficulties in arguing on a qualitative basis.

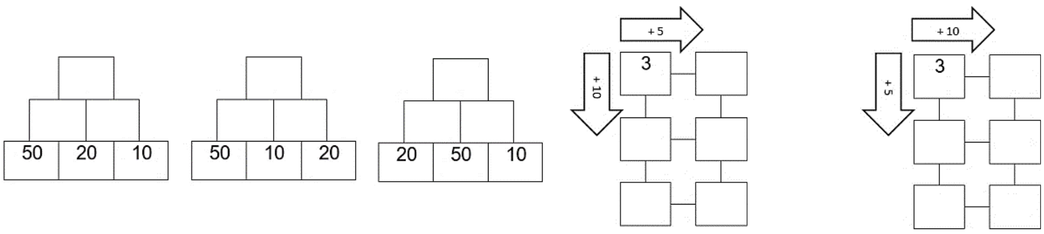

Figure 2: Task formats number pyramid (left) and numerical lattices (right)

RESULTS

(1) Wrong content or incomplete arguments

A reasonable difficulty in arguing are wrong initial observations and wrong or unconnected justifications for reasoned inferences. Incomplete arguments, in particular, arguments that leave questions open, could be observed most frequently, both in self-formulated (67.7%) and initiated warrants (44.2%) (see Jablonski & Ludwig, 2019).

(2) Lack of need for argumentation

The following example shows that a large proportion of the children, despite their professionally correct discoveries, do not feel the need for argumentation and that the discoveries would be unfounded without initiation.

I: When does the result [of a number pyramid] become the largest?

B1: If the largest number is in the middle.

I: Exactly. And is it always like this?

B1: Yes.

I: Why?

An analysis of all interviews shows that in 75.8% of the tasks without initiation no warrant is mentioned. In cases where there is a critical demand or a justification requirement, only 10.5% of the cases do not mention a warrant. Incomplete arguments can therefore often not be attributed to a lack of knowledge of content, but rather to a small need for argumentation (see Jablonski & Ludwig, 2019).

(3) Generalization by examples

It could also be observed that generally formulated conclusions were justified by means of example-based warrants, as in the following example, when the result of the numerical lattices is the greatest (see also "empiric argumentation" (Brunner, 2014)).

B2: If there, where it continues, the larger number is the arrow down.

[...]

B2: (*points on the result of the numerical lattices of task 2*) Because here you can see it well. Because here is the result of the first one bigger, because that is because it is going down and that is where you see the 10 and here the 4 and that is why the result is smaller than that.

Overall, in 43.75% of all tasks with a general conclusion, no or an exemplary warrant was coded. In addition, the step of generalization in elementary school age seems to be connected with the idea of being able to test the experimentally obtained statement on the basis of "all possible" examples.

> I: [...] Can you say that in general? Can you say that for every numerical lattices?
>
> B3: I think so, but I might have to try that first, but I'm not going to calculate all the number lattices here.

(4) Lack of certainty and persuasiveness

It is evident that arguments are linked to a degree of uncertainty (see "Modal Operator" (Toulmin, 2003)), regardless of whether the argument is substantively correct.

I: Why does this influence the result?

> B4: Because (...) I think because twice 20 plus 10 gives 50 and (...) maybe there something (…) no idea.

Therefore, the justifications do not seem to be brought forward in their original – convincing – function, but only out of the school context of the less authentic, initiated arguments (see Schwarzkopf, 2015).

(5) Minor importance for argumentation

Another reason for problems in the independent formulation of arguments may be the general significance of arguments and the process of arguing in the sense of a motivational problem.

> I: Maybe you can justify it again. It's not true because (...)
>
> B5: (...) oh, I do not like that when you have to justify something.

In addition to the lack of need for justification, there also seems to be a lack of will to justify discoveries. Hereby, former studies of mathematically gifted students’ intuition (e.g. Käpnick, 2010) might be relevant in order to explain redundancy of giving warrants.

(6) Language barriers

Various studies have already shown that language and mathematics are interrelated (Bezold, 2009).

> B6: [...] I do not know how to say that.

In the interviews too, various linguistic problems could be observed when it comes to expressing considerations intelligibly. Due to the complexity of the area, this will not be completed in the context of the article.

SUMMARY AND OUTLOOK

The results provide a differentiated insight into the qualitative difficulties of gifted children in arguing, which can be divided into dimensions of a contentual, methodological and motivational nature. Nevertheless, it can be assumed that the influences are connected and interrelated with one another in argumentation. Thus, an insufficient statement of reasons – insofar as this fact is known – can lead to a lack of certainty. Nevertheless, this is not a necessary criterion. Also substantively correct reasons were formulated with elements of uncertainty, and vice versa. The results may be one approach to fostering the development of argumentative skills and competences within the support of mathematical giftedness. Regarding the continual difficulties in the sense of (1) *Wrong content or substantial arguments*, a combination with the general support of mathematical giftedness is obvious. Nevertheless, one should especially make incomplete arguments subject of a discussion. In the context of (2) *Lack of need for argumentation* and (5) *Minor importance for argumentation*, it seems sensible to regularly emphasize the need for arguments, the significance and the convincing function of arguments through authentic "conflict situations" or "productive irritation" (e.g. Schwarzkopf, 2015). Furthermore, linguistic support measures in terms of (6) *Language barriers*, as well as the topics of structure, generalization and completeness of arguments make sense for all categories, but especially for (3) *Generalization by examples*.

Through a regular training of argumentative abilities, it might also be possible that (4) *Lack of certainty and persuasiveness* decreases through a normal implementation of the mathematical argumentation activity. Through this, the support of mathematical giftedness as a potential for outstanding mathematical performance is combined with a potential for outstanding argumentative competences.

The collected data is part of a longitudinal study on changes in the arguments of the gifted children within 1.5 years. Every six months, the students will be interviewed again, so that changes can be followed and the courses of changes can be classified. Within this changes, it will also be possible to see whether some of the described difficulties might decrease or disappear, whereas new difficulties might arise. During the conference presentation first results from the second data collection will be included as a basis for this discussion.

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EXPLORATION OF UNKONWN: A DIFFERENT APPROACH TO FOSTER MATHEMATICAL CREATIVITY

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Abstract. *In India, mathematical talent is nurtured by training students who have apriori shown an aptitude for problem solving techniques in mathematics typically under a time constraint. Many mathematicians argue that mere problem solving does not foster mathematical creativity. This article is about an ongoing experiment spread over 35 weeks (of which 20 weeks have elapsed) involving a group of 40 school students who were given opportunities to foster their creativity by exploring the unknown. The focus was to provide an environment and opportunity to discover mathematical concepts without teaching them formally and observe its role in nurturing creativity. Thus, these children learnt by constructing the mathematical concepts by themselves.*

Key words: Creativity, Pattern-observation, Exploration, Critical Thinking, Inquiry, Problem-solving.

INTRODUCTION

Understanding the process of mathematical creativity gained importance in early 20th century (e.g., Hadamard 1945, Wallas 1926). Surprisingly, most mathematicians are uninterested in analyzing and documenting the process (Ervynck 1991). Although documenting the above mentioned process will be useful, particularly for students interested in mathematical research, most mathematicians are reluctant to do so. Mathematics is as much an art as it is a science and important mathematical discoveries can be attributed to creative thinking among the mathematicians (Sriraman 2009). In this paper, we explore the possibility of fostering creativity among school students through problems that have an intuitive appeal, but the concepts have not been introduced in a formal manner to these students. Our objective is to highlight the possibility of fostering mathematical creativity among school students through the process of exploration of the unknown and provide evidence that the process results in nurturing of this particular mathematical ability.

Generally, mathematics is taught in schools through algebra and geometry with arithmetic as a building block and topics such as probability, graph theory, discrete mathematics, and indeterminate equations are considered to be college level topics, difficult for school students to grasp. However, we experienced that solving problems without a formal introduction encourages out of the box thinking and logical thinking in students. We observed that the students were able to apply the problem-solving strategies already learnt to completely new scenarios encountered even when the problems belong to a topic, they have never encountered. Our experience indicates that this approach of exploring the unknown fosters critical creativity, thinking skills, problem solving approach, and most importantly makes mathematics more interesting and meaningful. This approach can complement the inquiry-oriented mathematics instruction used to develop creativity (Silver 1997). Our main contribution is to demonstrate that exploring the unknown territories is important for nurturing mathematical creativity for the school students. We documented our approach and we are narrating the experience in the following sections.

THREE INCIDENTS

In April 2018, during a workshop on Combinatorics, we asked a question on Probability to a group of 7th and 8th grade students. The question was - If there are 10 rows in an airplane and 3 seats per row on either side of the aisle, what is the probability that two friends get the neighboring seats assuming their seats are randomly allocated by the airlines (consider aisle seats as neighboring)? Since most students participating in the workshop were introduced to elementary probability in their schools, the assumption was that this problem was of a medium level difficulty. As soon as the question was framed, a student asked, "What do you mean by probability?" The teacher explained by providing elementary examples of dice and coins. Taking the conversation ahead, the teacher continued, "What's the chance of you getting any particular seat in the plane?" The student replied that it's1/60. The teacher then asked, "If one person gets a particular seat, what's the chance that his friend will get the neighboring seat?" The boy replied, "Wouldn't it depend whether the seat the person has got is a window seat or a non-window seat?"[14] It wasn't expected that the student would solve it, but the teacher sat next to the 7th grader just to see how he worked through the problem. It took five minutes for the student to figure out the answer. His approach was, "There are two cases - I get a window seat and otherwise. There is 1/3 chance of me getting a window seat. And 1/59 chance that my friend will get the neighboring seat. There's 2/3 chance of getting a non-window seat and 2/59 chance that my friend gets an adjacent seat. Thus, the answer will be $\left(\frac{1}{3} * \frac{1}{59}\right) + \left(\frac{2}{3} * \frac{2}{59}\right)$." This was a stunning moment for us because we had never imagined that someone who isn't introduced to probability would solve this as the first problem in probability. In the past, we had posed this problem to many undergrad students and hardly 5-10% in a class who could obtain the desired answer.

Another incident happened in December 2018 when we introduced a problem on Conditional Probability to a group of 6th & 7th graders without *formally* introducing the concept of Probability. The question was, 'There are 6 switches out of which 3 are working. If you switch on any 3 switches randomly, what's the probability that all 3 are working?' The matter of interest was to see how the students work through it. In the previous lecture with the same group, a similar problem of mixing flavors of ice-creams from a given set of 6 flavors and creating new flavors was asked. Many worked it out through brute force and logical ways to list down all possibilities. But in this second session, within 90 seconds, a 6th grader came up to the teacher and said that there's 5% chance of such an event happening. The explanation he gave was impressive. He said, "There are equal number of good and bad switches. Therefore, the probability that we get the first switch as a good one is ½. Once we have taken out a good switch, the chance of getting the second one good is 2/5 and similarly the third one is ¼. Multiplying all the three probabilities we get 5%."

At another instance, a 9-year old solved a $\binom{n}{k}$problem without knowing Combinations. The problem was - In how many ways can an ant travel through the lines of a 8x8 chessboard from the bottom left corner to the top right corner if it can travel only north

[14] Such interactions are important to ensure that the students understand the problem on hand. In other words, it is a process of negotiating meaning rather than imposing fixed procedures (Bishop, 1985; Kozmetsky, 1980).

and east. The moment he saw the problem he said, 'I know how to solve this problem.' He didn't mean he knows a formula. He meant, by looking at the problem he was able to figure out a strategy to approach the problem. It took him about 30 minutes to work it out and he made some silly calculation error due to which the answer came to a difference of about 150 numbers which was a 1% error.

BRINGING 'OUT' CREATIVITY

The above-mentioned problems have an intuitive appeal. Hence, the students were able to solve some of these problems which are only introduced in high school or undergrad in many countries. The regular teachers of the students mentioned in the above incidents would mostly not even expose them to such questions thinking that it would be way beyond their abilities. That certain topics are meant to be introduced at higher grades, is a common belief. But our experience tells us otherwise. In many classes, we have introduced problems to students of middle and high school and have seen that they figure out a way *naturally* without being taught. That, we believe, is creativity in the making. When we stop pouring in, we allow things to come out. This highlights the necessity to invest in developing mathematical insight (Leikin, 2013; Singer, Sheffield, Freiman, & Brandl, 2016; Sriraman, 2009) because creative problem solving is correlated with mental flexibility (Star & Newton, 2009).

To foster creativity and mathematical talents, we started working with forty students from 4th to 9th grade split into five batches according to their ages. Each batch had a weekly session of 3-hour duration without any break. In most sessions, we gave problems involving mathematical concepts that students have not been introduced to and asked them to work on them, initially alone, and later together. During the preliminary sessions, most of them had no clue how to go about it because in their experience, they were given problems after teaching certain concepts but here they were getting problems where they had to think how to even start approaching the problem. But because they had to think how to solve this problem and probably because they also understood that answers weren't going to come from us, all of them were actively involved in working out the problems.

Inventions and accomplishments in mathematics are a function of creative talent and not mere traditional academic achievements (Milgram & Hong, 2009; Sriraman, 2005; Usiskin, 2005). Generally, students are given problems to solve and are seldom asked to create a problem on their own. They might come up with creative ways to solve the problem, but that need not enhance their ability to *create* a problem on their own.[15] For example, in one of the sessions, students were given a sheet of hexagonal grid and asked to frame problems on their own. In the beginning they wondered how to start, but soon they were out with problems such as - can we count how many hexagons will be there on a hexagonal board (see Fig #1), can we find patterns in the count as the layers of hexagons increase, can we generalize the result and arrive at a formula, can we prove or disprove the formula, can we do something similar for a triangular board (which will look something like a pile of hexagons) on a hexagonal grid, can we do the same on a triangular grid for triangles and hexagons, and so on. It needs to be noted that none of these

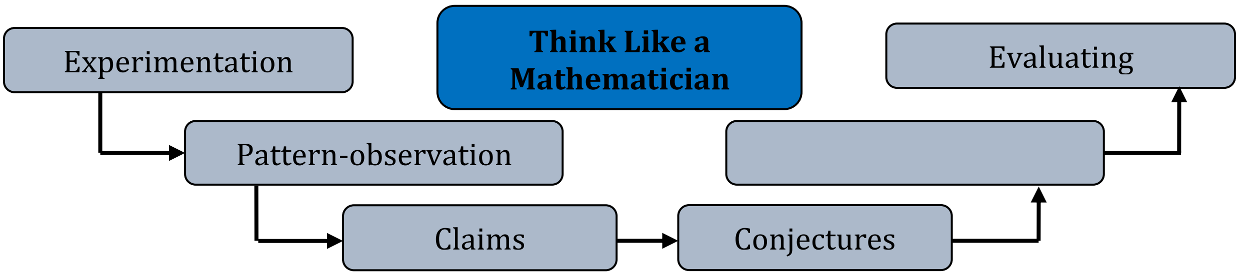
[15]Silver (1997) mentions generating new solutions for a mathematical problem to develop originality. However, we focus on providing opportunity to students to frame problems along with generating solutions to develop originality.

questions were given to the students, rather they created all of them and were more than thrilled about the fact. As soon as they were done, they wanted to figure out the solutions for all these questions. The passion and zeal for solving these problems were much more than what we could see in the problems that we gave them. And that was probably because they were *their* problems and not problems imposed by a teacher.


Fig #1: Hexagonal Board of side 6 units

It is well established that diversity is important for collective creativity (Alencar & Fleith, 2003; Leikin & Pitta-Pantazi, 2013). Even companies form teams with members having diverse skills to solve complex problems (Van den Bossche, Gijselaers, Segers, Woltjer, & Kirschner, 2011). These findings also extrapolate to creative outcomes in mathematics through synergistic benefits from collaborations among students because of different skills each of them had. For example, some were good with coming up with patterns and observations, while others were good with coming up with generalizations and claims. Those who were good with algebra and logic could argue to support a claim. Some of them were not good with any of these but they could think like a critic and find out counter examples for some of the claims. The point that needs to be noted here is that any research is an outcome of synergies generated by collaborative efforts of people with different skill-sets. Initially in the sessions they were individual players trying to compete against each other to get the correct answer first (even though we had never expected them to do that), but later they learnt to become good collaborators and achieved greater results than they would achieve individually.

THINKING LIKE A MATHEMATICIAN



Before the idea of a proof is introduced, somewhere in the middle school, students tend to take conjectures for granted. If they see a pattern, they feel it is true and the pattern is going to continue in that way.[16] To break this mindset, the students were given problems

[16] Arbona, Gutiérrez, Beltrán-Meneu, & Jaime (2017), for example, focus on geometric pattern problems to introduce linear equations for mathematically gifted students of grade 5 whereas we focus on critical thinking and reasoning to ensure that the pattern can be generalized or disproved

where it seemed that a pattern exists but in reality, it does not after a point. This forced them slowly to come up with some arguments to support their claims. Initially, all of them were observing samples and concluding that it would work for the entire population. Here, they were introduced to the difference between proofs in Sciences and proofs in Mathematics and this made them realize the importance and power of logical arguments. The regular process followed in the sessions was:

Given an experiment/data, first they need to come up with some observations/patterns. The second step is to come up with a claim and test it on a lot of cases. If the claim is true, it is held as a conjecture. It was of great interest to many students to try and disprove others' conjectures, and such a thing developed their critical thinking and articulating their thoughts with clarity. Finally, the task was to come up with a logical argument to prove the claim if it couldn't be disproved. Through this process, the students started thinking like a mathematician.

SCREENSHOT OF A SESSION

Just to give an idea of how a typical session went, below is one question that was explored in the sessions along with some snap shots of the findings by the students.

Question: Take any number with any number of digits, say *abc*. Sum up the cubes of each digit in *abc*. Iterate the process until you find something interesting. Analyze your findings. Come up with your claims. Prove/Disprove your claims.

Fig #2: First experiment with random numbers

Fig #3: Second experiment in a chronological order

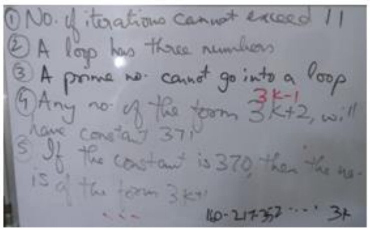


Fig #4: Claims

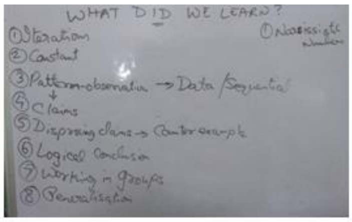


Fig #5: Students' feedback at the end of the session

It can be observed that in Fig #2, students were experimenting randomly but in Fig #3, they realized that a logical ordering for trials can help them arrive at a pattern faster. Also, the Fig #3 shows the collaborated efforts by all students who divided the task amongst themselves rather than working as individual players.

OUTCOMES/OBSERVATIONS

Patience and Perseverance: Over a period of twenty weeks, we could witness increase in students' patience and perseverance. They were willing to spend more than an hour on a problem in contrast to a few minutes on problems when they joined the course.

Creativity: When students were introduced to college mathematics without revealing the difficulty level of the problem, they exerted all their effort to solve them expecting that it is within their reach. They came up with very interesting approaches (sometimes, non-routine) to solve the problems which many a times are not seen in college students because college students often try to walk through the track that has been sketched in the class while teaching the concept and here the school students were trying to etch their own track.

Motivation: They had the motivation to solve problems because the students felt that they owned those problems.
Questioning and Reasoning: Students also increased the level of questioning and reasoning from the time they joined this program. Hence, most sessions ended up with an increased interaction among the students and not just between the teacher and students. Generally, students accept formulae from teachers without questioning. However, our students started questioning and critically evaluating anything presented to them.

Imagination and Visualization: There were instances where students had to stretch their imagination. For example, the students explored the topic of spherical geometry by changing a single axiom in Euclidean geometry that plane is not flat but a sphere. They then constructed conjectures and theorems by changing one axiom and some definitions. As it was not practically possible to do things easily on a sphere as much as it was on a plane, students visualized and communicated the abstract idea through arguments. Similar outcomes were experienced with problems involving tessellations and grids.

Abstraction: Abstraction is a very important skill and long-term goal of mathematics education (Wheatley, 1991). Students were able to abstract ideas from one problem and create new problems of similar type on their own. As a problem-solving strategy, they also started looking at earlier problems to check if the current one on hand could be solved using any ideas from earlier problems, thus developing their ability to abstract.

Articulating with clarity: Students find loops holes in another's definition and present counter examples. Hence, over the months they became more mindful of articulating definitions and arguments with clarity and precision. This is a necessary skill in both mathematics and science.

CONCLUSIONS

Based on our classroom experience and studies, we feel that students when exposed to topics from higher level mathematics when they are a clean slate, tend to solve problems more logically because their minds are not conditioned by a specific way of thinking. On the other hand, when students are exposed to a topic directly and asked problems based on what they have learnt, they do not push the boundaries of their mind and restrict their thinking to whatever they have learnt. This has been our observation when we have discussed topics from college mathematics with undergraduate students. We also believe that creativity and imagination in mathematics can be developed by giving students an arena to explore rather than giving them specific questions. It not only improves their mathematical abilities like abstraction, problem-solving, etc., but also improves qualities like patience and perseverance which are very much required to work through rigorous and challenging problems. These ancillary but important goals can be achieved when the teachers shift their goal from instruction in order to understand a topic to facilitation in order to imbibe certain abilities like creativity. By shifting the focus to a higher-order-thinking outcome, teachers would be able to develop abilities such as abstraction, creativity and problem-solving, which in turn are transferrable to other domains.

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SELECTION CRITERIA FOR STUDENTS IN A DANISH MATHEMATICS TALENT PROGRAM

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Abstract. *In this paper, we present the results of an interview study of Danish mathematics teachers' selection criteria regarding their selection of their most talented math students to participate in a two-year talent program, Talent² for Grade 5 students. Talent²is a development program for math teachers and talented math students, which focus on developing students' creativity and Mathematical Mindset. The results indicate that the math teachers have selected their talented students on the basis of either test results or the teachers' instincts. The teachers have not thought particularly about creativity as relevant criterion. The interview study also reveals that the talent students can be categorized in two groups: students that are generally result-orientated and achieving above average in all school subjects and students with higher level of creativity.*

Key words: creativity, mathematics, middle school, talent, selection criteria

INTRODUCTION

The reform of the Danish public school in 2013 is an ambitious bet that in these years is causing schools to undergo the most extensive changes in recent years. One of the purposes of the reform is to increase the proportion of the most talented students in Danish and mathematics year by year" (The Danish Government, 2018). In the wake of the reform, the ministry has offered the "The Fund for Talented and Highly Gifted Children" of 8 million Danish kroner (1.07 million €). The fund must support the development of targeted efforts in relation to the highly talented, underperforming and mistaken students (Danish Ministry of Education, 2016).

One of the development projects supported by the fund is called Talent². This project aims to guide teachers and prospective teachers in the teaching of students with a talent in mathematics. The participating mathematics teachers from four municipalities in the western region of Denmark were asked to select their most talented student in their Grade 5 math classes as the starting point of the two-year project. At the initial meeting, the teachers were introduced to a model of a talented student, which pointed out that a talented student cannot only be spotted by the time the student uses on a given task. One also needs to take into account the student's ability for contemplating, looking for patterns and being inquisitive and creative in the ways they solve tasks. This model builds on a Danish study made by Mogensen (2005).

Furthermore, in continuation of the development project, Talent² we have had opportunity to proceed with a research project, aiming to gain information about the participating students, their Mindset and creative thinking. We concentrate on two of the four municipalities, including nine participating math teachers and 37 math students. In the beginning, we randomly selected four of the nine teachers and their students to participate in our research. We contacted the school principals but we only got permission to collaborate with two of the four math teachers and eight talented math students.

In this paper, we will focus on the first phase of Talent[2]: the math teachers' selection of the participating students and on how the teachers describe these as the most talented students in their class and thereby the right candidates for the program.

CONCEPTUAL FRAMEWORK

According to Parish (2014), the talented students that are achieving great things are easy to spot. However, even though it can be easy to spot this group of talented children, there can be different criteria that teachers adopt to identify these students, different criteria for what the teachers understand as 'great things' in the math class. Further, there can also be introverted gifted students that are overlooked (Parish, 2014). In this paper, we will focus on the teachers' understanding of the characteristic of their most talented math students. In the context of the talent program, our research question is:

By what criteria did the math teachers in Talent[2] identify and select the talents students to the math talent program?

In order to answer the question concerning the selection criteria and the math teachers understanding of their talent students' performance in class, we wanted to focus on the role of students' creativity and relations between creativity and the students' high achievement in mathematics. We have chosen the Renzulli's three-ring-model to define and analyze the mathematically talented students (Renzulli, 2011). The model presents three elements: Above Average Ability, Task Commitment and Creativity. It is within the intersection of these three elements that we identify students as being talented.

The third criteria of creativity, however, remains a subject of discussion in mathematics. Sriraman, Yftian & Lee (2011) have defined creativity in mathematics both for professional and school level. When talking about creativity in our research project, we have chosen to focus on the definition for the school level:

> 1. the process that results in unusual (novel) and/or insightful solution(s) to a given problem or analogous problems, and/or
>
> 2. the formulations of new questions and/or possibilities that allow an old problem to be regarded from a new angle (Sriraman, Yftian & Lee, p. 121)

In both Talent[2] and our research project, there is a significant focus on the students' insightful solutions, their creative ways of seeing mathematics. In this case, we use the concept of mathematical mindset.

> When students see math as a broad landscape of unexplored puzzles in which they can wander around, asking questions and thinking about relationships, they understand that their role is thinking, sense making, and growing. When students see mathematics as a set of ideas and relationships and their role as one of thinking about the ideas, and making sense of them, they have a mathematical mindset (Boaler, 2016, p. 34)

METHODOLOGY

In this paper, we will present the preliminary results from our analysis of the semi-structured interviews with the math teachers that are responsible for the selection of eight talented students. The interviews had three themes:

- The teachers' selection criteria for the talented math students, participating in Talent2.
- The teachers' description of the students' math skills according to Renzullis three-ring-model.

- The need for differentiated math activities in the students' daily math classes.

In the research project, we have also interviewed and observed the eight talent students while solving different open math activities. These observations were then compared with the performance of a control group of nine students. The math teachers selected the students for the control group. They represent average math students from the same classes as the talent students.

The observations and the interviews with the math students had focus on three themes:

- The student's mindset. Growth or fixed mindset (Dweck, 2006).
- Flexibility and creativity, when solving math problems. Math problems selected from the categories Perception and Flexibility of thinking (Krutetskii, 1975).
- Mathematical mindset and creativity when solving open math problems. A math problem selected from YouCubed (Stanford Graduate School of Education, n.d.).

The students' interview provides opportunity to compare the teacher's description and understanding of the selected students to their performance in the students' interview/observations. However, in this paper, we will concentrate on the interviews with the teachers.

As the first part of this semi-structured interview, the mathematics teachers were asked to describe how they selected the students for the program. The math teachers had not been introduced to any models or definitions of talent in this first part of the interviews. The purpose was to find out what the main selection criteria according to the math teachers were, without the influence of external models. Therefore, when answering the question, the math teachers used their own definitions and understanding of talented math students.

In the second part of the interviews, we presented Renzulli's three-ring-model to the teachers and the eight talent students were analyzed and compared to the three categories for the model one by one.

THE TEACHERS' SELECTION CRITERIA

Three of the eight talent students were selected on the basis of their test results. The math teacher looked at the results from the National tests in mathematics as well as the MAT test (a Danish standardized math test, developed for each Grade level) where the selected students scored the highest results in their class. Four more students were chosen by their math teachers' 'common sense'. In these cases, the teacher simply knew which students was to be selected. After the information meeting with the program consultants and educators from VIA University College**,** one more student was selected. A student that according to her teacher have good understanding, is good explaining to the others and is able to take the mathematics to a higher level.

When we asked more about which properties could describe the students selected by the teachers' instinct, teachers mentioned criteria such as "the need to be noticed", independence, good overview and the ability to teach others. The first criteria, ”the need to be noticed”, is about all the resources the math teacher normally use with students with special needs, and according to the teacher there is often minimal time left for the talent students.

At the same time, independence is one of the both teachers' criteria, as one says it should not be necessary to give much help and the students should be able to read the tasks by themselves.

Above average ability

All eight students were selected to the program because of their above average ability. In six cases, these were confirmed by the results from the latest National tests in mathematics. The other two students have according to their math teachers a great ability, although they only score as average students in the tests. One of these students has difficulties with reading. This is reflected in his test results which indicates an above average score in number problems, but only average in text problems. According to his math teacher, the student has a good self-confidence in mathematics and do not use any learning aids such as reading programs to help in math tests. The other student is socially and physically very active. He has difficulties with concentration and motivation but when he gets started, the results are above the average.

Task commitment

According to the teachers, task commitment is the category that is the easiest to relate to all the eight students. However, the students demonstrate different kinds of task commitment. 4-5 of the students are generally committed to do well in school; their goal is to finish their tasks in time and do exactly what is needed to solve the problems chosen by the teachers.

The other group of 3-4 students shows their task commitment by their motivation to get deeper into the problem and make connection. In their working flow, they even have difficulties to stop and leave for the breaks.

> Teacher 2: She immerses herself, that is, quite wildly, she may like to sit in the break and work further, because if it is something she does not think she has done or wants to investigate more then she will, then she would like to use the break on it.

The math teachers also indicate that these students do not give up so easily.

> Teacher 1: Because he wants to find out, and he does not just hit the pitch straight away if he cannot figure it out in two seconds, he does not.

Creativity

The most difficult part for the teachers to analyze was their talent students' abilities within creativity. The teachers did not have a clear understanding or definition of creativity in mathematics. When presented with Sriraman, Yftian & Lee's (2011) definition of creativity in mathematics in school level, the teachers claimed there were some levels of creativity with all eight students.

Turning to the other group, which includes the student first chosen after the information meeting and the student that has problems with his concentration and, occasionally, motivation, the teachers were able to give some examples of students making systems and finding patterns, and according to the teachers, loving it.

As in the case of task commitment, the students' creative performance followed the two groupings. The former gr oup of students had their focus on finishing their task in time; there were no indications of creativity above the average students' performance. These students are by their teachers described as result oriented and their first priority is to finish all assignments in time, no more or no less.

When comparing the math teachers' free description of their selected students and their description with help from Renzulli's three-ring-model, we see that in terms of the creativity, the teachers are uncertain about five out of eight students. The one student selected by his teacher after the introduction meeting with the consultants from the Teacher Education is, according to the teachers, is the student that shows the most characteristics associated with the center of Renzulli's three-ring-model.

DISCUSSION

The results from the interviews indicate that the participating math students could be categorized under two different groupings. The first group could be described as the result oriented students that are ambitious and want to do well in school, including mathematics. These students perform above average in tests, have an ability with numbers, complete all assignment and are according to their teachers' general good students. For these students, it is very important to do well.

> Teacher 2: She does not feel confident if she does not get it right the first time. She is able to immerse herself in the task, but she would rather finish earlier than immerse herself in the task.

The other group might be more interesting for our research project when we in next phase of our research will focus on the students' attitude and performance when solving math problems. In this group, we have students who, according to their teachers, exhibit a higher level of creativity than the first group. Their task commitment do not only include being finished with their task, but more often also indicate a motivation to investigate further and make connections and systems. In some cases, they do not even want to stop and leave the math class for a break. In this group, we also have one student that is underachieving his potential ability. His lack of concentration and motivation in math class could be interesting to investigate further to get further understanding of what the math teacher could do to motivate and support his development. This student is one of the students that the Danish Ministry of Education's and Talent^2 aimed at recruiting into to talent development program, as this would present an opportunity to help them find the motivation and further develop their potential.

In this case, we only have small population of eight students out of 37 participating students from Talent^2. In the case of three of them, the math teacher focus on test results, and have not given much thought to creativity as one of the aspects. When describing the students' performance in class, the teachers tend to focus more on work ethics:

> Teacher 1: He is attentive during review on class and he likes to work. If he has a small complaint, he will sometimes be a little bit too quick in terms of doing things properly. But otherwise, he's very positive and reasonably smart.

Therefore, the students might have more creative potential in math than is otherwise visible in the daily classroom. In this case it would have be interesting to investigate the students development through their work in Talent^2 to see if new approaches to mathematics with more open math activities and discussions that support the talents students creative thinking could contribute to the students' creativity in mathematics.

Even though we do not have the opportunity to follow the students' development and learning in the program, in the phase two of our research we will be able to compare the eight students from the talent program to nine students, selected randomly from the same classes and the math talents. All students have been given the same questions about their mindset, inspired by Dweck (2006), and open problems to solve in pars of two students in sessions of 45 min. Here, we aim to investigate indications of relationship between task commitment and a growth mindset as well as developing creativity and indications of mathematical mindset (Boaler, 2016) when solving mathematical problems, in cases where the students approach math with sense making and intuition.

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MATHEMATICALLY GIFTED STUDENTS' REFLECTIONS ON USING HISTORY OF MATHEMATICS IN MATHEMATICS CLASSROOM

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Abstract. *The aim of this study is to examine the reflections of mathematically gifted students on using history of mathematics in mathematics classroom. This case study examined 12 fifth grade mathematically gifted students' opinions about using history of mathematics in mathematics classroom. Students' opinions were taken with video evaluation forms after they had watched videos on biographies of mathematicians. The findings of the study revealed that mathematically gifted students reflect their opinions about using history of mathematics in a positive way. Students opinions were categorized into three sub-themes: learning more information, demanding further research and motivation to achieve.*

Key words: Differentiated instruction, history of mathematics, mathematically gifted students, using videos.

INTRODUCTION

Gifted children are those who demonstrate outstanding ability in one or more domains such as intellectual, artistic, creative, leadership and/or in specific academic areas such as language, science and mathematics when compared to their same age peers (National Association for Gifted Children, 2005). They differ in their learning in terms of learning new concept quickly, rich memory, working with more complex and abstract ideas, transferring their learning into other settings and intense curiosity (Davis & Rimm, 2004). For mathematically gifted students, the fundamental differences are in their cognitive abilities, motivation behaviors, self–adequacy, emotional control and learning style (Rotigel & Fello, 2004). Teachers should provide appropriate educational opportunities that will meet the needs of gifted students. When they plan the instruction, individual learning profile of gifted, affective needs, abilities, multiple intelligence and learning preferences should be considered (Agnes& Kainose, 2017). However, few teachers consider these differences in their classroom practice (Guild, 2001).

Differentiated instruction can be an effective tool to address diversity of learners' needs. Meanwhile, it supports multiple intelligences in classroom and various learning styles. Differentiated classroom environments provide opportunities for students to do their best (Tomlinson, 2003). In the differentiation of mathematics instruction for gifted students, it is important that the subjects of famous mathematicians are included in the lesson such as their lives, contributions to life, personal characteristics, thinking and working habits (Yevdokimov, 2007). D'Ambrosio (1995) argues that by investigating mathematicians' cognitive activities in the past, students can better understand the role of human minds in constructing mathematical knowledge, and accept that they too can produce mathematical ideas that are original. Hence, individuals can become aware of their own abilities and learn about how to maximize their own potential. This allows gifted children to have a deeper knowledge of mathematics discipline and to become aware of their abilities (Ozyaprak, 2016). For that reason, history of mathematics can be an effective tool to differentiate mathematics instruction for gifted students.

Jankvist (2009) identifies the role of the history of mathematics in mathematics education. It increases motivation through generating interest and excitement and it decreases intimidation through the realization that the mathematics is a human endeavor and that its creators struggled as they do. Jankvist (2009) also mentions history as a pedagogical tool that can give new perspectives and insights into material and even can serve as a guide to the difficulty students may encounter as they learn a particular mathematical topic. On the other hand, there are wide range of possible ways of integrating history in mathematics classroom, including primary sources, worksheets, mechanical instruments, textbook studies, visual means etc. (Fauvel & van Maanen, 2000, pp.214-232). However, in light of the characteristics of gifted students, technology can be an essential tool in providing education to address the specialized needs of gifted students (Morgan,1993). Technology-enhanced environments provide affordances that support and create intrinsically motivation for gifted students (Housand & Housand, 2012). Accordingly, this study aims to examine mathematically gifted students' reflections on using history of mathematics in mathematics classroom.

METHODOLOGY

This case study examined 12 fifth grade mathematically gifted students' opinions about using history of mathematics in mathematics classroom. The study was conducted in a private school where only gifted students attended in Turkey. Wechsler Intelligence Scale for Children-Revised (WISC-R) score (above 120) and aptitude tests were used for identification of gifted students in this school. The participants of the study were determined as mathematically gifted. It is one of scales commonly used in intellectual assessment in our country for many years. It is an intelligence test, which provides Verbal and Performance Intelligence Quotient (IQ) scores obtained separately, and it consists of two sections, each containing six subtests (Mahone et al., 2003). Eight videos on famous mathematicians' lives were selected by mathematics and social science teachers and integrated into mathematics instruction. In the selection process, key characteristics of mathematics curriculum for gifted students such as providing interdisciplinary connection, context with greater dept and higher level of complexity (Johnson, 1993) and topics in fifth grade mathematics curriculum were considered. For example, Euclid video was watched during the teaching of geometry concepts and Gauss video was watched during the teaching of sum of successive numbers. Students' opinions were taken with video evaluation forms developed by the researchers and categorized into sub-themes by using item based in-dept content analysis. Sample questions from the video evaluation forms are as follows: What do you learn after watching the video? What do you notice from watching the video? What is the most interesting thing while watching the video? Does the video about biography of mathematician in mathematics lesson contribute to you? If your answer is yes, how?

FINDINGS

All written responses sentence by sentence and identified words or phrases that described the students' opinions were categorized. The findings of the study revealed that using history of mathematics via technology enhanced environment affected students' opinions in a positive way. Students' opinions are grouped under three sub-themes: learning more information, demanding further research and motivation to achieve.

Learning more information.

When students' reflections on using history of mathematics were examined, they stated that they learn more information about famous mathematicians, their lives and contributions. They thought that videos are more entertaining, realistic and detailed and expressed their opinions as follows:

> Student 1: I learned that Omer Khayyam is not only famous mathematicians but he is also poet and Islamic scholar. Student 2 also stated that x notation is found by Omer Khayyam.
>
> Student 4: I learned that Pascal invented first numerical calculator and found the interior angles of triangle is equal to 180 degree. Moreover, I learned atmosphere pressure changes depending on the altitude.
>
> Student 5: After watching the videos, I learned that Pythagoras is known as the father of the numbers and he make a connection between mathematics and music.
>
> Student 12: I learned that Euclid wrote the book of 'Elements' which is comprehensive work in mathematics and made many contributions into mathematics.
>
> Student 8: I learned that Pascal found a planet and used triangle for different purposes like probability Student 3 added that Pascal directed Geometry and had a talent in mathematics when he was a young age and made inventions.

Students' opinions reflected their interest in learning more information about occupations of famous mathematicians and their contributions not only in mathematics but also in other fields. Moreover,

> Student 9 expressed opinion as follows: I learned that $\sqrt{2}$ cannot be written as the ratio of two integers proved by the student of the Pythagoras. However, Pythagoras did not accept the work of students.

Demanding further research.

According to analysis, students expressed their opinions about demanding extra research on mathematics branches such as numbers, geometry and algebra. For instance,

> Student 2: I want to learn algebra and geometry in more detail. I want to make a research about rational, irrational and prime numbers.
>
> Student 5: I wonder about such questions after watching videos. How are the further geometry topics? What are famous scientists in history of geometry who I do not know? What are the contexts of books in the field of geometry?
>
> Student 10: I want to learn more about axioms and proof methods

Findings revealed that they want to learn more advanced topics for their grade level. More specifically, some students explained their curiosity in deeping their mathematical understanding on topics.

Also, student 11 wanted to research about mathematics and other interdisciplinary topics such as history of telescope, 4 satellites of the Jupiter. Student 3, 6 and 8 wanted to research about law and the literature. They also wonder about the process they experienced to invent something. For instance,

> Student 9: How Pascal invent first numerical calculator at a young age?
>
> Student 3: What are difficulties Pascal experience when making invention?
>
> Student 12: What are the ancient mathematical terms and why learning such terms are difficult?

Motivation to achieve.

The motivation to achieve sub-theme not only leads individuals to pursue work they perceive to be valuable; it also prompts them to compete with others (Covington, 2000). This drive may come from an internal or external source. Achievement motivation is intrinsic when it is sparked by an interest or enjoyment in the task itself. It is organic to the person, not a product of external pressure. Achievement motivation can be instead extrinsic when it comes from outside the person. Common sources of extrinsic motivation among students are rewards like good marks.

According to students' reflections, using history of mathematics in mathematics classroom motivated gifted students to achieve something. They are influenced by mathematicians' life to aspire it. For instance,

> Student 4: I did not give up until doing something after watching video of Galileo Galilei. He supported his postulate "the earth rotates around sun"
>
> Student 2: Mathematics can help us in every field of life such peace and war. I learned mathematics contributes to humanities.
>
> Student 11: Atatürk invited German scientists to Turkey writing a letter to Einstein. They observe the situation of Turkey and contributed to the development of Turkey. This motivates me to make an effort for my country.

DISCUSSION

This study examined mathematically gifted students' reflections on using history of mathematics in mathematics classroom. When students' reflections were taken into consideration, it can be said that the opinions of the mathematically gifted students on using history of mathematics in mathematics classroom are positive because the videos address their unique needs by getting their attention and arising their interest. It is clear that using different method in mathematics classroom for differentiation can increase the ratio of students who gain more knowledge and whose attention was drawn to the lesson. One of the reasons why they have positive opinions about using history of mathematics may be due to gifted students' needs of in-depth knowledge and challenge (Rotigel & Fello, 2004). Also, students' reflections support the idea that cultural understanding and replacement supported by history helps to humanize mathematics education. In addition, using history of mathematics may help mathematically gifted students clarify "why" and "how" questions arising in mathematics classroom (Aydemir & Isıksal-Bostan, 2015). The findings of this study are consistent with the study of Yevdokimov (2007) that using history of mathematics in mathematics instruction help students to enhance their understanding of different ideas and theories, to motivate them for further learning and also to indicate the richness of human activities in mathematics. Mathematically gifted students as bright of society can be directed to more research and can contribute the development of society. Using technology enhanced history of mathematics make them more curious about research and invention. Thus, this study could be beneficial to address the unique needs of mathematically gifted students by getting their interest, learning style and learning profile.

The participant of this study is limited with fifth grade mathematically gifted students. Further research can be conducted with other middle school graders or primary school students. The reflections of mathematically gifted students can be comparable with other students on using history of mathematics. Furthermore, the effectiveness of other methods such as mathematical textbooks, historical problems and primary sources when integrating history of mathematics can be investigated in further studies.

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STUDENTS' CONCEPTIONS OF MATHEMATICAL CREATIVE THINKING AND CRITICAL THINKING IN STEM PBL ACTIVITIES

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Abstract. *Mathematical creative thinking and critical thinking are essential to be successful in STEM-related post-secondary academic and career pathways. Therefore, the aim of the present study was to investigate the development of students' conceptions of mathematical creative thinking and critical thinking through STEM PBL activities. A group of 39 students in grades 7-12 participated in the intervention (STEM PBL activities) and completed pre- and post-surveys concerning their conceptions of mathematical creative thinking and critical thinking. Results showed that students' conceptions about their mathematical creative thinking and critical thinking were statistically significantly higher after engaging in STEM PBL activities. The findings from the current study support the importance of STEM PBL for the development of students' positive conceptions toward creative thinking and critical thinking in mathematics.*

Key words: Creative thinking; Critical thinking; Science, Technology, Engineering, and Mathematics Project-Based Learning (STEM PBL)

INTRODUCTION

Critical thinking and creative thinking are considered essential higher-order thinking processes in mathematical educational contexts. In particular, the ability to effectively utilize both types of thinking is a desirable skill in all aspects of science, technology, engineering, and mathematics (STEM) fields because it allows individuals to better develop their STEM-related knowledge (Klimovienė, Urbonienė, & Barzdžiukienė, 2006; Leikin & Pitta-Pantazi, 2013). In fact, the development of both thinking abilities becomes a promising strategy helping to increase learning effectiveness in STEM–related subjects (Klimovienė et al., 2006). Students who have developed mathematical creative thinking and critical thinking abilities within STEM contexts will be ready to enter post-secondary STEM pathways (Bicer et al., 2018). The Organization of Educational and Economic Development (OECD, 2009) has emphasized the importance of the development in both creative and critical thinking as the essential skills for 21st century competencies that need to be encouraged by educators. In particular, creative and critical thinking are interrelated and complementary aspects of higher-order thinking process and thus need to be developed together.

Creative Thinking

Creative thinking is the process of "the creation or generation of ideas, processes, and experiences" (Klimovienė et al., 2006, p. 80). Students who use and develop creative thinking in their learning environments become sensitive to relative/non-relative information, generate hypotheses, and obtain results (Torrance, 1966). Furthermore, research findings indicated that students who were more likely to think creatively possessed higher levels of critical thinking (Klimovienė et al., 2006). In the process of thinking creatively, there is a confluence of knowledge, intelligence, style of thinking, personality, motivation, and learning environments (Sternberg & Lubart, 1995) that boosts individuals' imaginative, constructive, and generative creative thinking (Bailin, 1987). Therefore, teachers need to provide effective instructional methods that will stimulate students' development of creative thinking.

Critical Thinking

Critical thinking is an analytic process that demands higher-order thinking to achieve a desired outcome within a given framework or context. Critical thinking is defined as the ability to understand arguments, analyze and connect relevant information, and evaluate acquired information according to criteria or logic in order to make reasonable decisions or judgements (Halpern, 2014). In the processes of critical thinking, students were required to use inductive and deductive reasoning, interaction of divergent phases of problem solving, implement creative processes, and adopt motivational and affective functions (Miele & Wigfield, 2014). Therefore, critical thinking can be considered as a multidimensional construct. Additionally, it involves assessing information in real-world situations and sociocultural contexts to make life decisions (Bailin, 1987). Therefore, classroom environments that provide real-world situations could foster students' critical thinking development.

An interest in adopting innovative educational settings and activities to promote creative and critical thinking has increased among educators in mathematics education. Research has indicated that innovative educational approaches in terms of learning environment and activities encouraged students to make use of creative and critical thinking (Fairweather & Cramond, 2010). In particular, STEM project-based learning (PBL), has been shown to be an effective educational tool for innovative teaching and learning, encouraging students to communicate and cooperate with their peers in a real-world context (Capraro & Slough, 2013). Therefore, STEM PBL is likely to improve students' creative and critical thinking. If STEM PBL activities effectively foster creative and critical thinking, the impact would be apparent through students' STEM-related academic achievement (Arum, Roksa, & Velez, 2008) and their STEM-related career aspirations (Bicer et al., 2018). However, little has been researched about possible innovations in teaching and learning, and whether STEM PBL activities nurture students' mathematical creative thinking and critical thinking (Chan, 2013). Therefore, the purpose of the present study was to investigate whether students' conceptions toward their mathematical creative thinking and mathematical critical thinking are developed through engagement in STEM PBL activities.

METHODOLOGY

In the present study, we used a nonrandomized quasi-experimental design that had a single treatment to examine students' development of creative and critical thinking through STEM PBL activities. Participants were middle and high school students (N=39) who were enrolled in a two- week STEM camp held on a university campus in the southwestern United States during the summer of 2018. The camp consisted of a diverse group of students in terms of gender (female=18, male=21), grade (7^{th}=1, 8^{th}=6, 9^{th}=7, 10^{th}=7, 11^{th}=7, 12^{th}=11), and ethnicity (African-American=3, Asian=4, Caucasian=20, Hispanic=8, others=2, missing=2). We designed STEM PBL activities (e.g., coding, microcontroller, drones, 3D printing, etc.) that were then implemented during the intervention to provide participants with collaborative, hands-on activities to complete given projects. The STEM PBL topics were taught on a daily basis over the course of 10 days.

We created instruments for assessing students' creative thinking consisting of 11 items (e.g., "Crazy sounding ideas can lead to something.", "Productive change is important to a business. New ideas foster change. Therefore, new ideas are important to business.") and critical thinking that consisted of 5 items (e.g., "When a theory, interpretation, or conclusion is presented in class or in the readings, I try to decide if there is good supporting evidence.", "I often find myself questioning things I hear or read in this course to decide if I find them convincing.") The survey questions consisted of a five-point Likert scale from 1 (Strongly disagree) to 5 (Strongly agree). The Cronbach's alphas as the reliability coefficients were .67 for creative thinking and .86 for critical thinking. The instrument was administered as a pre-test before the camp began and as a post-test after the camp concluded. Each administration was completed within 10-15 minutes.

SPSS 24 was used for statistical analyses. Paired-sample *t*-tests were used to determine the mean differences of students' creative thinking and critical thinking between pre- and post-tests. Descriptive statistics including Hedge's *g* and associated 95% confidence intervals (*CI*s) were also reported.

RESULTS

Descriptive statistics of the pre- and post-test performances are presented in Table 1. In particular, Table 1 shows means, standard deviations (*SD*), range of scores (minimum and maximum), and 95% *CI*s (lower and upper limits) of students' performance on the pre- and post-tests of mathematical creative thinking and mathematical critical thinking.

		Mean	*SD*	95% *CI*s	Range	
					Min	Max
Creative thinking	Pre-test	32.62	3.92	[31.39, 33.85]	23	39
	Post-test	34.59	4.17	[33.28, 35.90]	27	44
Critical thinking	Pre-test	26.41	3.80	[25.22, 27.60]	16	33
	Post-test	27.62	4.65	[26.16, 29.08]	19	35

Table 1: Descriptive data of students' performance in mathematical creative and critical thinking

The results from the present study revealed that students' creative thinking and critical thinking were developed through participating in STEM PBL activities (see Figure 1). In creative thinking, the mean score of the post-test performance was higher than the mean score of the pre-test performance. The results from the paired-sample *t*-test showed that the mean differences between pre-test and post-tests were statistically significant (t=1.97, df=38, p<.05), and the Hedge's g effect size of mean difference was g=0.64. In critical thinking, the mean score of the post-test performance was higher than the mean score of the pre-test. The results of the paired-sample *t*-tests showed that the mean differences of pre-test and post-test were statistically significant (t=1.21, df=38, p<.05), and the Hedge's g effect size for the mean difference was g=0.39.

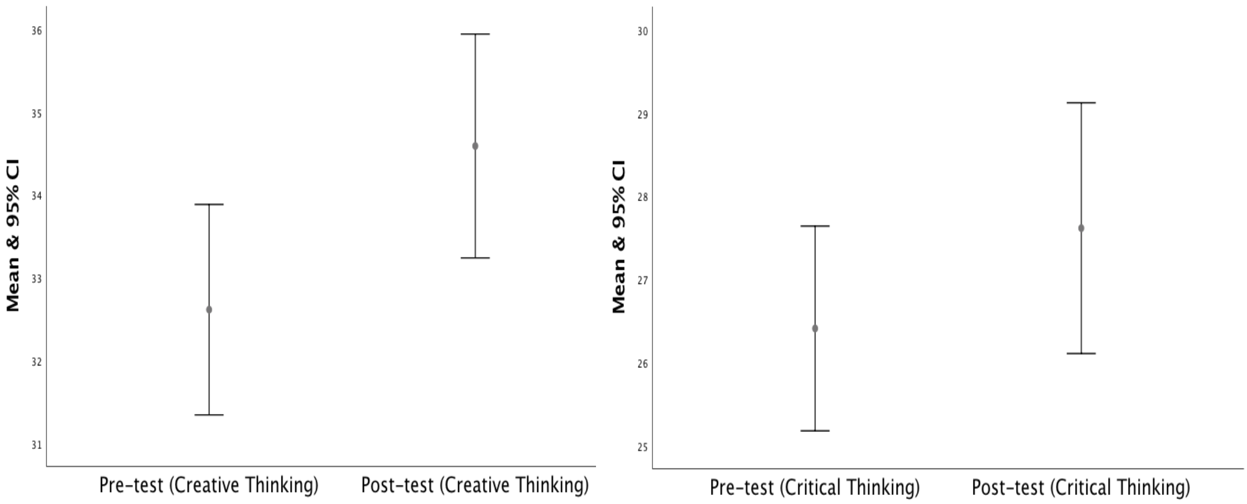

Figure 1. Means & 95% CIs of Pre-& Post-tests in Creative & Critical Thinking

DISCUSSIONS

A central goal of contemporary mathematics education is to improve students' higher-order thinking skills, and the notions of creative thinking and critical thinking provide a focus for this effort. Mathematical creative and critical thinking abilities consist of more than isolated technical skills. Creative and critical thinking need to be presented within real and dynamic contexts (Bailin, 1987). For the present study, we assumed STEM PBL as one effective instructional method to enhance both types of thinking skills. We investigated the development of students' conceptions toward their mathematical creative and critical thinking through STEM PBL activities. The preliminary results showed that STEM PBL activities positively influenced both creative and critical thinking conceptions of students.

These findings point out that STEM PBL is crucial in any attempt to improve the thinking skills of the students both within interdisciplinary areas in real-world situations and with respect to collaboration with their colleagues. In the processes of STEM PBL, students encouraged to sharing their thinking and cooperate with their peers to solve the problem in a real-world context (Capraro & Slough, 2013). These processes might encourage the students to mathematical reasoning, divergent thinking and multidimensional approaches in sociocultural contexts, which is highly related to the development of mathematical creative and critical thinking (Bailin, 1987; Miele & Wigfield, 2014).

Teaching students to develop their higher-order thinking must be a priority in schools today. Schools are responsible for providing opportunities to go beyond learning simple subject matter knowledge and achieve students' higher level of skills and habits in their creative and critical thinking. The extent to which the development of thinking abilities depends on the commitment of the educators inside and outside of mathematics classroom. In particular, the findings of this study about the positive impact of STEM PBL on nurturing students' mathematical creative and critical thinking supports the importance of adopting STEM PBL activities in mathematics classrooms and the necessity of professional development for mathematics teachers.

Teachers' high level of pedagogical content knowledge toward STEM PBL may lead the good quality of teaching and learning contexts and boost students' development of mathematical creative and critical thinking.

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TASKS THAT ENHANCE CREATIVE REASONING: SUPPORTING GIFTED PUPILS IN INCLUSIVE EDUCATION SYSTEMS

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Abstract*: This paper represents the first stage of a wider study on how to support mathematically gifted pupils' creative reasoning in inclusive education systems. The study focuses on gifted pupils in Norway, which has a one-track education system where pupils are organised in heterogeneous (mixed-ability) classes with few opportunities to meet other gifted pupils. In the present study, we observed gifted pupils' reasoning when working collaboratively in both heterogeneous and homogeneous learning environments. The purpose of this paper is to discuss the creative potential of one of the tasks used in our study. We present preliminary findings from the study, focusing on pupils' written products from collaborative work in homogeneous groups. The analysis identified a variety of methods used to solve the tasks, and we used the pupils' written products from work on one of the tasks to exemplify this variety.*

Key words: gifted pupils, learning opportunities, rich tasks, inclusive education

INTRODUCTION

Inclusive education is an approach that embraces diversity and aims at promoting equal learning opportunities for all pupils. The present study was carried out in Norway, where inclusive education is an overarching principle and classrooms are organised as heterogeneous (mixed-ability) learning environments. Adapted education is a central principle in inclusive education systems and refers to the idea of providing appropriate learning opportunities so that all pupils can achieve their potential (Fasting, 2013). However, the reality is that many pupils are not given learning opportunities that correspond to their learning potential. In fact, there is a category of pupils that is almost invisible in inclusion debates—gifted pupils (Smedsrud et al., 2018; Tirri & Laine, 2017). In this paper, we address the challenge of supporting mathematical learning opportunities for gifted pupils in inclusive schools.

Inspired by Lev and Leikin (2017), we consider creativity to be a central aspect of gifted pupils' mathematical learning opportunities. Sriraman (2009) defined creativity as "the ability to produce novel or original work" (p. 15). Creativity can be explored in multiple ways. Lev and Leikin (2017) explored mathematical creativity by employing multiple solution tasks (MSTs). The main characteristic of MSTs is that pupils are "explicitly required to solve a mathematical problem in different ways" (Lev & Leikin, 2017, p. 228). Rich tasks are examples of tasks that can be solved by multiple solving processes (Ashline & Quinn, 2009). Such tasks have been found to support mathematical learning opportunities in inclusive classrooms for all students (Boaler & Sengupta-Irving, 2016; Mellroth, 2017).

The main purpose of our study is to explore mathematically gifted pupils' reasoning with respect to creativity. Lithner (2017) considered the task and the pupils' mathematical competence to be critical components of their creative reasoning (see Figure 1). He proposed two crucial design principles for tasks that aim to promote creative reasoning: the pupils should not know the solution method in advance and the task should not be too difficult for the pupils to solve (p. 941). Therefore, as the first analytical step in our wider study on this topic, this paper focuses on the creative potential of one of the tasks used in the study. For this purpose, we use pupils' written work from working on one of the tasks to discuss and exemplify the task's creative potential in light of Lithner's (2017) conceptual framework for designing tasks that enhance creative reasoning. The following research question guides this paper: *What creative aspects can be found in gifted pupils' written products from collaborative work on rich tasks?*

METHODS

The study presented in this paper involved four children (aged 13–15) who were following a national programme for gifted pupils (involving learning centres for STEM talents in Norway). Participation in this programme is offered to pupils who excel in mathematics and natural sciences, and those who participate meet four times during the school year for two days at the learning centre in their region. The Norwegian education system does not allow fixed-ability grouping, and pupils in grades 1–10 are never held back a year (The Education Act, 1998, § 8-2). The national programme for gifted pupils in Norway started as a pilot in 2016 and is part of the government's strategy for supporting talents and high-performing pupils.

To pursue our interest in gifted pupils' mathematical reasoning, we interviewed the pupils about how they experienced their school situation. We then video-recorded them while they worked collaboratively on rich tasks in two different learning environments: heterogeneous and homogeneous groups. In the first setting, they worked together with peers in their regular classroom. In the second setting, they worked together with other gifted pupils at the learning centre in their region. This allowed us to study their mathematical reasoning in both heterogeneous and homogeneous small groups. The analysis of the pupils' reasoning is still in progress. In the present paper, we report on the creative potential of one of the tasks that the pupils worked on in the homogeneous small groups. To do so, we examined written products from their work at the learning centre.

The purpose of video-recording the collaborative work was to gain insight into c*ollective solution spaces.* According to Levav-Waynberg and Leikin (2012), collective solution spaces can be a helpful "tool for examining the mathematical knowledge and creativity of participating students" (p. 78). According to the gifted pupils in our study, they are offered more mathematical learning opportunities at the learning centre than in regular classes. This claim is in line with Webb's (1980) description of group behaviour in heterogeneous groups. By exploring collective solution spaces in both heterogeneous and homogeneous settings, we seek a better understanding of gifted pupils' learning opportunities in inclusive mathematics classrooms. Inclusive settings are characterised by a diversity of pupils that perform at different levels, with different learning styles and different interests. While we note the ongoing discussion on how to group pupils according to ability (e.g., Francis et al., 2017), the research presented in this paper seeks to contribute to the strand of research focusing on pupils' learning in light of creative reasoning (e.g., Haavold, 2011). Because rich tasks have been found to promote mathematical learning opportunities for pupils performing at different levels (Boaler & Sengupta-Irving, 2016; Mellroth, 2017), we used these kinds of tasks to gain insight into gifted pupils' learning opportunities in both heterogeneous and homogeneous settings.

In analysing pupils' written products, we draw on Lev and Leikin (2017), who emphasised the importance of multiple solutions, and Lithner (2017), who claimed that it is easy to design tasks that require creative reasoning but difficult to make the tasks easy enough for pupils. Figure 1 provides an overview of factors influencing pupils' reasoning (Lithner, 2017). In relation to this figure, Lithner (2017) suggested that pupils' reasoning (2) depends on the pupil's mathematical competence (1), the task properties (3) and the teacher's task-solving support (4). In the present paper, we address task properties (3) in a learning environment where we assume that the pupils' mathematical competence (1) and task-solving support (4) are similar/homogeneous for all participating pupils.

Figure 1: Overview of pupils' reasoning (Lithner, 2017, p. 938).

The task we used in this paper to exemplify and discuss the creative potential of rich tasks is presented in Figure 2. This task can be solved in multiple ways and can therefore potentially offer different challenges depending on the pupil's mathematical competence. Because the task is not in any of the participating pupils' mathematics textbooks, we assumed that most of the participating pupils would not have worked on this task in advance. Therefore, we assumed that the task met the two principles proposed by Lithner (2017): the pupils should not know the solution method in advance and the task should not be too difficult for the pupils to solve (p. 941).

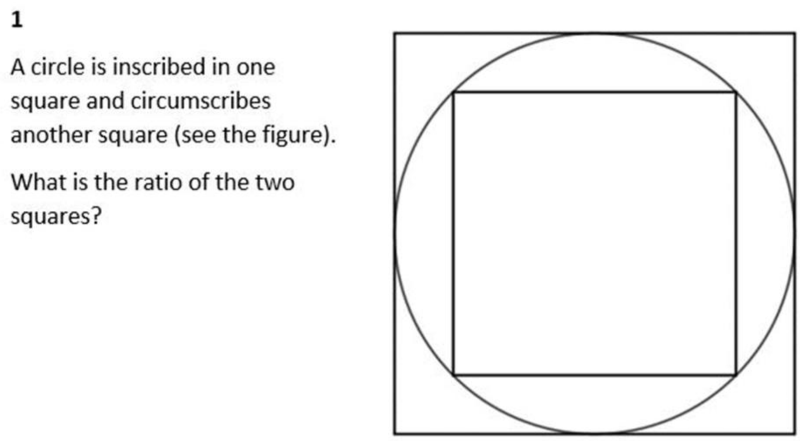


Figure 2: The task examined for the purposes of this paper.

RESULTS

Our conversations with the pupils prior to their group work on rich tasks indicated that they saw themselves as auxiliary teachers when they worked in heterogeneous groups. They said that their learning process tended to be put on hold because they had to give explanations to their fellow pupils in the group and motivate them to work in a focused manner. These experiences are in line with the findings reported by Webb (1980) on group behaviour in heterogeneous groups.

The analysis of the groups' written products revealed four different solution methods (see Figure 3). In the following, we address these solution methods with respect to the two design principles proposed by Lithner (2017).

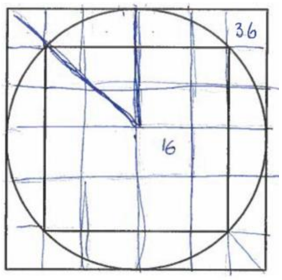
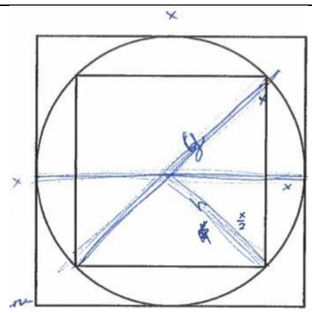
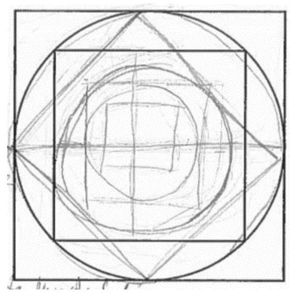
Method 1	Method 2	Method 3	Method 4
Measure one side of each square and multiply it by itself to find the area of both squares. Then compare the areas of the two squares.			

Figure 3: The four solution methods used by the pupils.

The analysis of the solution methods demonstrated that the first two methods presented in Figure 3 were based on measurement and proportional reasoning. For both methods, the pupils first expressed the area of each square by a number and then compared the two numbers to find the ratio. In method 1, the pupils measured the actual length of the sides and used a formula to calculate the areas. In method 2, the pupils used non-standard units of measurement to express the areas. Methods 1 and 2 are both numerical methods. The pupils used methods they knew in advance to find the areas, but they used these methods in a new setting (given that they did not know the task in advance).

The analysis of solution methods 3 and 4 indicated that both methods involved dynamic aspects of pupils' generalisation. Both methods had a mental transformation as a starting point. More precisely, the pupils transformed the figure by rotating the inner square 45°. We interpret this rotation as the pupils' argument for justifying that the inner square's diagonal equals the sides of the outer square. From this point onwards, however, solution methods 3 and 4 took different pathways. While method 3 developed into a symbolic generalisation based on Pythagoras' theorem, method 4 consisted of informal reasoning leading to non-symbolic generalisation.

Figure 4 presents the complete written product from the group that came up with method 4, including both the drawing and the text. The reasoning for method 4 was not possible for us to grasp from the drawing or the text alone. Neither of these components can be fully understood without considering the other. While the drawing demonstrates that the pupils rotated the inner square, this was not easily understood from the written text, which explains that "we combined the outer circle parts with the biggest square's corners." The text continues by suggesting that "this formed a right-angled triangle that had the size of 1/8 of the biggest square." Analysis of the text alone did not help us to understand the pupils' justification for this suggestion. However, considering the text and the drawing as intertwined parts of the group's reasoning allowed us to better understand the appropriateness of the arguments behind method 4.

We combined the outer circle parts with the biggest square's corners. This formed a right-angled triangle that had the size of 1/8 of the biggest square. The total size of the triangles equals 1/2 of the figure and the rest of the space equals the smallest square. The smallest square equals half of the biggest square.

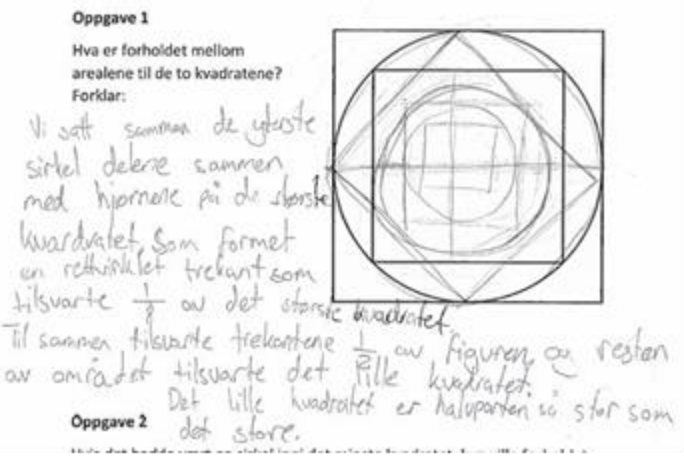


Figure 4: Method 4—the pupils' written answer.

DISCUSSION

In this paper, we have examined the written products from the pupils' collaborative work on a rich task using Lithner's (2017) conceptual framework for designing tasks that enhance creative reasoning. The interpretation of these written products is the first analytical step of our study focusing on how to support mathematical learning opportunities for all pupils in inclusive education systems. The interviews with the gifted pupils showed that they experienced limited learning opportunities in their regular classrooms, occupying the role of auxiliary teachers rather than learners. These experiences are important parts of the pupils' self-reported learning opportunities. However, the experiences cannot be considered detached from research reporting that heterogeneous learning environments are appropriate for promoting learning opportunities for all students, regardless of their level of achievement (e.g., Boaler & Sengupta-Irving, 2016). The Norwegian school model protects inclusive education, and legislation prescribes that pupils should receive an education adapted to their abilities and aptitudes (The Education Act, 1998, § 1-3). This means that all pupils are placed together in classes regardless of academic level, while at the same time they must receive academic challenges suited to their abilities. Considering the potential dichotomy between the pupils' experiences and the research findings, there is a need for a better understanding of how to promote adequate learning opportunities for gifted pupils in inclusive settings. One of the aims of our study is to contribute to a better understanding of how all pupils, including the gifted, can be offered adequate learning opportunities in inclusive settings. This has formed the rationale for our investigation of written products from gifted pupils' collaborative work on rich tasks in heterogeneous and homogeneous groups.

The investigation of pupils' written products revealed four different solution methods. While some of the written products seemed to be directly based on solution methods familiar to the pupils from their regular classes (methods 1 and 2), others seemed to involve more novel aspects (methods 3 and 4). This is not to say that one solution is more creative or sophisticated than the other. Because the pupils had different years of schooling (they were between 13 and 15 years old), they could not have been expected to have the same mathematical tools to solve a task. An important next step for better understanding the novelty of each solution method will be to combine the analysis presented in this paper with an analysis of pupils' interactions within each group. Deciding whether or not a written product is based on creative reasoning is difficult when only partial aspects of the pupils' solution processes are analysed. We have already experienced this in our analysis of the texts and drawings as isolated parts, and we assume that our analysis of the pupils' communication will contribute to an even deeper understanding of the creative aspects of the pupils' solving processes.

CONCLUSION

This paper has addressed the creative potential of a rich task used in our study focusing on mathematical learning opportunities for gifted pupils in inclusive education systems. We analysed pupils' written products from collaborative work in homogeneous groups to gain insight into the task's creative potential. The analysis showed that the pupils used four different solution strategies. According to Lithner (2017), a task has the potential to enhance creative reasoning when it is not too difficult for the pupils to solve and the pupils do not know the solution method in advance (p. 941). The four solution methods exemplified the creative potential of the task in line with these principles: it could be solved in multiple ways and it was not too difficult for the pupils.

What the gifted pupils said during the interviews prior to working in small groups showed that they did not think that group work in the regular class provided them with academic challenges suited to their abilities. However, the written products presented and discussed in this paper paint a more nuanced picture of the effectiveness of homogenous groups. In terms of Lithner's model (Figure 1), the four products were produced as answers to the same task (3) in the same learning environment (4) by pupils that were considered gifted (1). Despite these similarities, the analysis revealed a variety of creative aspects in the written products.

These findings demonstrate the complexity of using tasks that "require" creative reasoning. Although the task we have presented and discussed here aimed to enhance the pupils' use of creative reasoning, it did not lead to all the pupils using creative reasoning to solve it. Based on the homogeneity of the pupils and the learning environment, we consider this an important first step for better understanding the creative aspects of gifted pupils' reasoning in both heterogeneous and homogeneous groups. In fact, we observed the same learning processes in a homogeneous learning environment that the pupils in our study reported from their regular classroom experiences, namely, that one task could be solved in multiple ways (some of which were claimed to be more elegant than others). This finding raises the question of how homogeneous a homogeneous group really is.

The final conclusions with respect to gifted pupils' creative reasoning will emerge when we have completed the microanalysis of pupils' communication based on the framework developed by Lithner (2008).

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EXAMINING PRIMARY SCHOOL TEACHER-SUPPORT TOWARDS MATHEMATICALLY GIFTED LEARNERS IN SOUTH AFRICA

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Abstract*: This paper reports on teacher preparedness in supporting mathematically gifted learners in mainstream classrooms. Empirical studies in South Africa show that such classes seem not to present conducive learning environments for gifted learners. It is against these observation that this study aims at exploring the support given by foundation phase teachers to mathematically gifted learners in Motheo and Xhariep districts' primary schools of Free State province. A Hundred and five teachers completed questionnaires and their principals were interviewed from twenty selected schools. The results show that teachers lack training which disadvantages the gifted learners to perform to their full potential. This paper concludes by recommending the continuing emphasis on teacher training in gifted education at higher institutions as well as in-service training at school levels.*

Key words: mathematically gifted, teacher preparedness, mainstream, foundation phase, gifted education

INTRODUCTION

In all classrooms, learners have diverse learning needs that due to failure to support and responded to would lead to barriers toward learning. Children with outstanding talent are viewed to perform or show the potential at remarkably high levels of accomplishment when compared with others of their age, experience or environment. Therefore, every teacher is expected to recognise such children in his or her class. Studies conducted in the United States toward professional development for teachers of mathematically promising students, reported students' highly significant progress (Singer, Sheffield, Freiman & Brandl, 2016). Furthermore, the inclusion of Teacher Preparation Standards in Gifted and Talented Education for colleges and universities (Singer *et* al., 2016), indicated teacher professional development to handle these mathematically promising students effectively. Consequently, this preparation of teachers entails the needed teacher competencies of gifted students as knowledge of student needs for an example, the development of a differentiated curriculum with multiple resources (Chamberlin and Chamberlin in Singer *et* al., 2016). However, empirical studies indicated that mainstream classrooms seem not to present ideal environments for gifted students in a practical context to develop to their full potential (Mhlolo, 2017). On the other hand, Blömeke and Delaney (2012) emphasized the importance of a teacher's professional motivation and self-regulation that enable her to define her professional objectives, decide on appropriate strategies in achieving them and apply such in various situations. Thus, teachers' competence is expected beyond the content knowledge. Yet, previous research indicated that teachers struggle to interpret tasks and identify their educational potential (Stahnke, Schueler & Roesken-Winter, 2016). Similarly, in South African schools gifted learners are still not receiving adequate support due to lack of teacher training particularly in catering for such learners' needs (Mhlolo, 2017).

THEORETICAL MODEL THAT CENTRALISES THE TEACHER IN DEVELOPING GIFTS INTO TALENTS

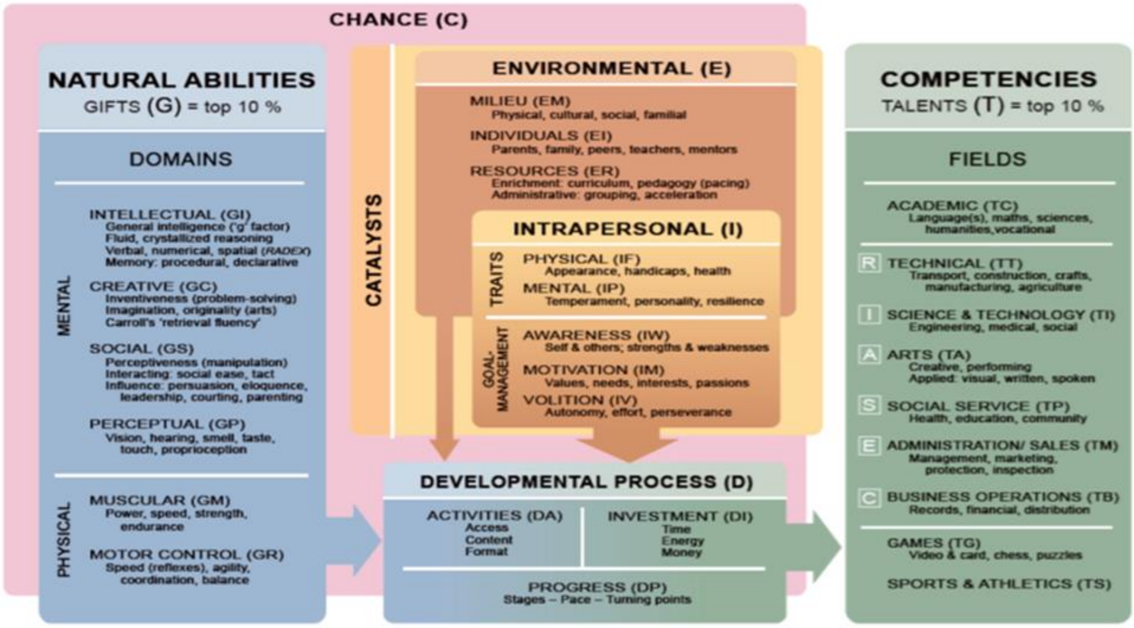

Figure 1: Gagné's Differentiating Model of Giftedness and Talent (DMGT) (Gagné, 2015)

This paper adopted Gagné's (2015) Differentiating Model of Giftedness and Talent (DMGT) as its theoretical framework, Figure 1, above. Gagné demonstrates that students' natural abilities which are gifts, emerge from mental and physical domains. The model demonstrates that Gifts need to be nurtured and developed systematically into competencies known as talents through developmental process by different catalysts. Gagné's (2015) model sensitizes researchers about the developmental process of the student's potential that could be affected negatively or positively by environmental and intrapersonal catalysts. Consequently, if the developmental process is affected negatively, it becomes poor and result in hindering the gifted student to become talented. Such student's inherent gifts will go to waste but if nurtured and developed, such potential (gifts) will reveal his or her talent. This framework is relevant to this study because the focus is on the teacher and the principal as environmental catalysts who manipulate all the resources in teaching and learning environment. Gagne's model suggests the strategies that teachers can use in teaching gifted learners. Such strategies include differentiation and grouping enrichment of all subject matters in regular curriculum (Gagné, 2015). Furthermore, the contribution of individuals determine the success in teaching all learners including the gifted learners. So, if teachers are not trained in gifted education, gifts won't be translated into talents. Freeman (2011) asserted that teachers intuitiveness and inspiration can sport and nurture talent. Given this, the researchers aimed at examining teachers support toward mathematically gifted because they are part of environmental or developmental process as catalysts.

MATERIAL AND METHOD

This study used a survey research design that is frequently used in educational research, the investigator selects a sample of subjects and administers a questionnaire or conducts interviews to collect data (McMillan & Schumacher, 2014). The rationale for using survey research is that it is very popular in education for its versatility, efficiency and generalizability. Cohen, Manion and Morrison (2011) indicated that the purpose of the interview is to gather data in surveys or experimental situations. The structured interviews were conducted with the purpose of obtaining the present perceptions of activities, feelings, motivations, future expectations or anticipated experience and, to verify and extend information from participants or researcher (McMillan & Schumacher, 2014). The researchers conducted the face-to-face interview for principals to elaborate on experiences that had helped or hindered their developmental support toward mathematically gifted learners. A questionnaire was used to obtain information from participants (McMillan and Schumacher, 2014) in order to examine the support of teachers toward mathematically gifted learners. So, the researchers used a Likert scale that had been one of the most commonly used tools that measures people's attitudes and to indicate a range of responses to the survey (Maree, 2007). Therefore, 105 foundation phase mathematics teachers who taught mathematics in grades 1 to 3, were purposively sampled to complete the given 3-point Likert scale questionnaires. The purposive sampling was used for this study for its relevance as the researchers were interested in mathematics teachers only. The foundation phase teachers were chosen because many high-performing students lose ground from elementary to middle school.

RESULTS AND DISCUSSION

Table 1 below presents the results on the first question of this study: To what extent do teachers support mathematically gifted learners in mainstream classrooms? This question generated three sub-questions: Did you receive training on how to teach gifted learners? Do you feel competent enough to teach gifted learners? Do you think Higher Education Institutions should include content on gifted education in their courses?

	Agree (%)	Neutral (%)	Disagree (%)	Total
Training received	30	50	20	100
Teaching competence	20	50	30	100
Inclusion of Gifted Education	90	6	4	100

Table 1: Teacher preparedness in teaching the gifted learners.

The first sub-question above aimed at investigating the acquired teacher training in regard to gifted education. This was important as the researchers wanted to establish if the result would yield the support that mathematically gifted learners need in order to learn to their full potential in mainstream classrooms. Table 1 shows that 30% of teachers agreed to have received training to teach gifted learners. The other 50% were neutral on being trained to teach gifted learners. The remaining 20% disagreed to have undergone such training. This suggests that the majority of teachers require teacher training in gifted education.

The second sub-question aimed at establishing whether teachers confidently regard themselves competent to teach gifted learners in their classes. The results indicate that 20% of teachers claimed to be competent in teaching gifted learners. The other 50% of teachers were neutral while 30% disagreed to be competent in teaching gifted learners. This could imply that the majority of teachers need special training and or in-service training in regard to being prepared to teach gifted learners in mainstream classes.

The last sub-question wanted to establish whether or not teachers saw it fit that the content on gifted education be introduced as higher Institutional course and as well as at school level as pre-service training. The results indicated in Table 1 that the majority of teachers, 90%, were of the view that gifted education should be introduced as a course content at higher institutions. The 6% appeared to be neutral and 4% disagreed on such opinion. This implies that teachers are in need of gifted education at higher institutions in order to be well prepared to teach gifted learners in their mainstream classes.

The following discussion is based on the results in regard to teacher preparedness in order to support the gifted learners during teaching and learning. Firstly, 30% of participants agreed to have been trained to teach gifted learners in mainstream classes. Then, 20% of teachers claimed to be confident to teach gifted learners in such classes. Lastly, 90% of participants were of the opinion that higher institutions should offer gifted education as the course content. These results confirm previous findings in South Africa that teachers do lack training particularly in gifted education (Oswald & de Villiers, 2013). Similarly, (Mhlolo, 2017) asserted that gifted learners are still not receiving adequate support in mainstream classes due to lack of teacher training particularly in catering for these unique learners' needs.

In regard to the interviews, the responses are presented using the acronyms below:

> I: What do you do as a principal to support ongoing staff development opportunities that provide information and strategies for teaching mathematically gifted learners?
>
> PF: I encourage educators to join mathematics bodies such as AMESA, Hey Maths, and programs for Professional Learning Committee (PLC) to empower one another and to learn other practices from colleagues.
>
> PG: The school does not do anything about mathematically gifted learners. It is only the expanded opportunity and then they get bored but up to so far the workshops are in place for educators to enrich their minds, even the department is helping us in that regard to see to it that educators are well trained in terms of trying to make mathematics easy for them to be able to teach learners effectively. But generally so, for gifted learners as the school, we are not doing anything till this far.

The responses above are examples of 20 principals who develop their teachers through workshops, PLCs and conferences such as AMESA and studying further for enrichment. It was also mentioned that teachers learn from one another collectively as Mathematics teachers to teach learners effectively. However, learners are given expanded opportunities that tend to bore them as indicated by PG above with no effort of supporting gifted learners specifically. The results are similar to the findings of the principals interviewed by (Oswald & de Villiers, 2013) who indicated that the gifted child would always be neglected without the necessary support as the focus is on improving pass rate for everyone.

CONCLUSION

Although the results indicated a bit of awareness of gifted education among teachers and principals, gifted learners are still not supported according to their needs. The minority of teachers claimed to have received training in teaching gifted learners, whereas (Mhlolo, 2014) asserted that the 15 Sub-Saharan countries do not offer teacher training specifically for teachers of gifted and talented students. As a result, the interpretation of this minority of teachers (30%), could be based on their own perception of identifying gifted learners in their classrooms. Therefore, participants of this study are not ready to teach gifted learners in mainstream classes. The results also show that few teachers indicated that they were competent to teach gifted learners. This raised the concern and it wants us to recall what (Mhlolo, 2014) said about countries which do not offer teacher training in regard to gifted education. The results also want us to recall what (Oswald & de Villiers, 2013) said about South African teachers who were interviewed in respect to gifted education. The principals' views on the support they provide to teachers to teach mathematically gifted learners, were indication of some trying to find support through Mathematical bodies and competitions or conferences. However, some were simply waiting for such support in order to meet these precocious learners' needs. If gifted education is continuously neglected Galton's elitist philosophy will be perpetuated where he indicated that giftedness is not for black people and the poor (Galton, 1869). Therefore, this paper concludes in emphasizing inclusion of gifted education at school level and higher institutions in order to support the gifted learners to perform to their full potential.

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STEM PROJECT-BASED LEARNING ACTIVITIES: OPPORTUNITIES TO ENGAGE IN CREATIVE MATHEMATICAL THINKING?

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Abstract. *Prior research has shown the use of science, technology, engineering, and mathematics (STEM) project-based learning (PBL) activities in a mathematics classroom increases students' interest in mathematics and fosters students' creative thinking abilities. During the summer of 2018, one- and two-week residential STEM camps were held at a university in the southwest region of the United States for 7th - 12th grade students (n=49). Students took STEM PBL courses involving hands-on learning within real-life scenarios. A quasi-experimental design was used to determine how well students' beliefs about mathematics, interest in applying mathematics, and beliefs about creativity in STEM fields could predict their interest in applying creativity in STEM fields. Results indicated that the variables were strong predictors of students' interest in applying creativity in STEM fields and ultimately pursuing a career in STEM.*

Key words: Beliefs, Creativity, Mathematics, STEM, Mathematics Application, Creative Thinking

INTRODUCTION

Science, technology, engineering, and mathematics (STEM) fields are expanding, and employers in the STEM sector are looking for innovative candidates. However, some students believe STEM fields are mundane and uncreative and, therefore, may not pursue these fields (Valenti, Masnick, Cox, & Osman, 2016). Mathematics is a foundational component of the other three STEM disciplines; therefore, educators need to determine methods to illustrate the connection between mathematics and creativity. For the purpose of this study, creativity is defined as being innovative or "discovering something not already known" (Nadjafikhah, Yaftian, & Bakhshalizadeh, 2012, p. 290). One method that can be used to demonstrate the link between the two is the implementation of STEM project-based learning (PBL) activities. These hands-on activities provide students with opportunities to engage with mathematics content in which they identify innovative solutions to problems based on real-world scenarios. The research question answered through the current study was the following: How do students' beliefs about mathematics, interest in applying mathematics, and beliefs about creativity in STEM fields predict students' interest in applying creativity in STEM fields?

Beliefs about Mathematics

Students' beliefs about mathematics have been extensively examined in prior research. Due to the variety of studies, there are numerous definitions of what mathematical beliefs measure. For the purpose of this study, beliefs about mathematics will be defined as a set of ideas or opinions about mathematics that shape students' experiences and actions (Brown, 2016). These mathematical beliefs are formed based on students' prior experiences with mathematics (Jacobson, 2017). Several researchers have argued that mathematics beliefs can be changed; however, they have also found that some students have strong beliefs toward mathematics that are difficult to alter (McDonough & Sullivan, 2014; Stylianides & Stylianides, 2014; Sun, 2018).

Overtime, students' mathematics beliefs become more strongly related to their future desire to apply mathematics (Davis-Kean et al., 2008). Therefore, it is important to engage students early in their academic careers with hands-on, real-world experiences that promote positive beliefs about mathematics.

Use of STEM PBLs to Promote Mathematics and Creativity

Mathematics is embedded in all sciences; thereby implementing STEM PBLs affords students the opportunity to look for creative solutions, as well as, promoting mathematical thinking and creativity. Students generally view mathematics as a useful but hard discipline (McDonnell, 2014). While some students believe mathematics is the simple memorization of rules and procedures, other students see it as a creative discipline (Schoenfeld, 1989). By integrating mathematics concepts into hands-on activities situated in real-world scenarios, students may begin to see the link between mathematics and creative thinking. Specifically, through the use of STEM PBLs, students are provided opportunities to find innovative solutions, gain perspectives, and use their creativity to solve real-world problems (Oner, Nite, Capraro, & Capraro, 2016). Brainstorming multiple solutions to real-world problems permits students to combine and enhance their STEM content knowledge and creative thinking (Land, 2013; Madden et al., 2013; Mayasari, Kadarohman, Rusdiana, & Kaniawati, 2016; Valenti et al., 2016). Cultivating this type of creative thinking, through the use of STEM PBLs, in the classroom will promote the application of mathematics.

METHODOLOGY

A quasi-experimental design was used in the present study to understand how well students' beliefs about mathematics, interest in applying mathematics, and beliefs about creativity in STEM fields could predict their interest in applying creativity in STEM fields. The design of the study consisted of multiple pre- and post-tests. Data were analyzed using descriptive statistics, 95% confidence intervals (CIs), Cohen's d effect sizes, t-tests, and multiple regression. For the purposes of this study, an effect size larger than 0.25 standard deviations was considered practically important. The American Psychological Association (APA, 2010) considers CIs to be the best-reporting strategy. This statistical analysis provides a range based on observed scores and represents the probability that 95 out of 100 obtained CIs from the same population will contain the population parameter (Huck, 2008). Effect sizes and CIs are suggested in order to assess the practical importance of the study and support meta-analytic thinking by providing future researchers the ability to extract data across studies to develop benchmarks.

Participants

The participants (n=49) were 7th – 12th grade students (male=24) who attended a one- or two-week residential STEM camp at a southwestern United States (U.S.) university during the summer of 2018. Of those students, eight were in 7th or 8th grade, 15 were in 9th or 10th grade, and 26 were in 11th or 12th grade. The demographic background of the students included the following: African American (2), Asian (6), Hispanic (10), Caucasian (26), and other (5). Most of the students were from Texas (45), 2 were from other U.S. states, and 2 were international students.

Setting

The STEM summer camps have open enrollment, allowing camps to fill on a first-come, first-served basis. Therefore, participants who enroll in the STEM camp usually already possess some interest in STEM fields. During the summer of 2018, there were various camp options available. Students could enroll in a general STEM summer camp or STEM specific camps, such as Aerospace Engineering, or Coding Infectious Diseases. All of the camps included STEM PBL activities. In the general STEM summer camps, students engaged in projects related to a variety of topics (e.g., microcontrollers, coding, physics, discrete mathematics, structures, etc.), toured various STEM labs on campus, and attended panels of STEM professionals. In all of the courses, instructors implemented mathematics into the STEM PBL activities to provide students with real-world scenarios that allowed them to think creatively to solve a problem. This approach was used to encourage students to see connections between mathematics and creativity.

Instruments

Two instruments were administered during this study: *Student Attitudes toward STEM (S-STEM)* survey (F.I.E. Innovation, 2012) and *Science, Technology, Engineering, Arts, and Mathematics (STEAM)* survey. The *S-STEM* survey measured students' beliefs about mathematics, science, and engineering and their interest in pursuing careers in these fields. For the purpose of this paper, only the 10 Likert-scale mathematics items were utilized. from the *S-STEM* survey. The ratings for the scale ranged from strongly disagree (1) to strongly agree (5). Three of the questions were negatively worded (i.e., Math is hard for me.) and were reverse coded for the data analyses. The *STEAM* survey contained 5 Like-scale items that were used to measure students' beliefs about creativity in STEM fields and their interest in applying creativity in STEM fields. The ratings for the Likert-scale items ranged from strongly disagree (1) to strongly agree (4). Because the ratings were different between the two instruments, the researchers multiplied the results to maintain proportionally equivalent scores across the two instruments. This process resulted in a range of scores from 5 - 20. Both instruments were administered prior to each STEM camp and at the conclusion of each camp.

An exploratory factor analysis was conducted using the 15 items, and the results indicated four main constructs (see Table 1). Beliefs about mathematics measured students' beliefs or opinions about their ability to do mathematics. Mathematics application measured students' desire or interest to apply mathematics in the future. Beliefs about creativity in STEM fields measured students' beliefs or opinions about STEM courses requiring creative thinking. Creativity application measured students' desire or interest to use their creativity in STEM fields.

Construct	% Variance Explained	Chronbach's α	Sample Items
Beliefs about Mathematics Ability	40.67%	0.89	I can get good grades in math. I am the type of student to do well in math.
Mathematics Application	22.69%	0.83	I would consider choosing a career that uses math.
Beliefs about Creativity in STEM	16.95%	0.85	I believe STEM courses and careers require a lot of creativity.
Creativity Application	12.16%	0.53	I would like to have opportunities at camp to use my creativity on one or more projects.

Table 1: Factor Analysis Results and Sample Items

RESULTS

Results from the surveys were analyzed using descriptive statistics, 95% CIs, Cohen's *d* effect sizes, *t*-tests, and multiple regression results. The purpose was to determine the impact that beliefs about mathematics, interest in applying mathematics, and beliefs about creativity in STEM fields could predict students' interest in applying creativity in STEM fields.

The descriptive statistics (see Table 2) provide a summary of the means, standard deviations, and CIs for the pre- and post-scores on the four constructs: beliefs about mathematics, mathematics application, beliefs about creativity in STEM fields, and creativity application. These results reveal a decrease in mean scores from pre- to post- in all constructs, except mathematics application. The effect size for the means from pre- to post- on mathematics application was d=0.26. The results from the paired sample t-test showed the mean difference between pre- and post-scores on mathematics application was statistically significant (t=2.09, p=0.04). No other statistically significant results were noted from pre- to post-scores.

Construct	Pre (1)/ Post (2)	$\bar{x}$	SD	95% CIs Range
Beliefs about Mathematics Ability	1	17.03	2.60	(16.28 – 17.77)
	2	16.84	2.91	(16.00 – 17.67)
Mathematics Application	1	17.20	3.19	(16.28 – 18.11)
	2	17.90	2.13	(17.29 – 18.52)
Beliefs about Creativity in STEM	1	16.53	2.79	(15.73 – 17.33)
	2	15.77	3.19	(14.85 – 16.68)
Creativity Application	1	15.37	2.21	(14.74 – 16.01)
	2	15.20	2.86	(14.38 – 16.03)

Table 2: Descriptive Statistics of Students' Beliefs about Mathematics Ability and Creativity

Multiple regressions were conducted to determine how well beliefs about mathematics, interest in applying mathematics, and beliefs about creativity in STEM fields could predict students' interest in applying creativity in STEM fields (see Table 3). The three variables accounted for 53% of the variance. Beliefs about creativity in STEM fields and interest in applying mathematics had a positive statistically significant impact on students' interest in applying creative thinking in STEM fields. However, positive beliefs about mathematics had a statistically significant negative impact on students' interest in applying creative thinking.

Construct	B	SE	β	t
Beliefs about Mathematics Ability	-0.44	0.11	-0.45	-3.91**
Mathematics Application	0.48	0.16	0.35	2.93*
Beliefs about Creativity in STEM	0.46	0.10	0.52	4.79**
R^2	0.54			
Adjusted R^2	2.01			

*$p<0.01$, **$p<0.001$

Table 3: Multiple Regression Results of Students' Interest in Applying Creativity

DISCUSSION

STEM education is one of the main foci in today's schools because of the growing need for skilled workers in STEM fields. The purpose of this study was to investigate the impact that students' beliefs related to their mathematical ability, interest in applying mathematics in the future, and beliefs of creativity in STEM fields could predict interest in applying creativity in STEM fields. The use of STEM PBLs in all camp classes encouraged multiple mathematical solutions and creativity (Land, 2013; Madden et al., 2013; Mayasari et al., 2016; Oner et al., 2016; Valenti et al., 2016). For instance, students in the physics class designed an innovative fidget spinner requiring them to calculate the different forces that were impacted when changing aspects of their design. Two goals of the physics class were to design (1) the fastest and (2) most creative fidget spinner. This highlights how instructors integrated mathematics content and creativity within their STEM PBLs.

Overall, data indicated that STEM summer camp significantly impacted students' interest in applying mathematics in the future. This could be due to the fact that students began to see how mathematics is integrated in real-world scenarios. Furthermore, beliefs about creativity in STEM fields and interest in applying mathematics were positive predictors for students' interest in applying creative thinking in STEM fields. Students who had higher beliefs about creativity in STEM fields and an interest in applying mathematics in the future, were also interested in applying creativity in STEM fields. This could be attributed to the use of STEM PBLs in the various STEM classes which promoted mathematical and creative thinking. Interestingly, positive beliefs about their mathematics abilities negatively predicted students' interest in applying creative thinking. Therefore, students who had more positive beliefs about their mathematics abilities were more likely to not feel the need to apply creative thinking in the future. This could be due to the fact that students may believe seeking creative solutions are unnecessary if they can follow traditional methods of solving problems. However, this finding highlights the disconnect between mathematics and creative thinking (Valenti et al., 2016). Two of the three factors investigated were accurate predictors for students' interest in applying creative thinking.

Implications for this study support the use of STEM PBL activities to promote mathematical and creative thinking and students' future desire to apply mathematics. Increasing opportunities for students to think creatively and engage in mathematical activities will help improve students' beliefs and interest in pursuing STEM careers and degrees, particularly that of mathematics. Furthermore, educators should continue to look for methods to improve students' beliefs about creativity in mathematics and STEM in order to engage more students in STEM fields. Encouraging creativity in a STEM setting, through STEM PBL activities, will ultimately lead to the pursuit of a STEM career.

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ANALOGICAL TRANSFER AND COGNITIVE FRAMING IN PROSPECTIVE TEACHERS' PROBLEM POSING ACTIVITIES

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Abstract. *The ability of making analogical transfers is an important skill in mathematics and beyond. The paper reports on the capacity of a group of prospective mathematics teachers to pose analogical problems within a project. It seems that most prospective teachers face difficulties in transferring deeper structural elements of the initial problem to their newly posed problems. The difficulties are related to an obvious lack of experience of communicating mathematical tasks, but their comments reveal also a superficial level of mathematical competence, level made better visible through this type of project activity than through a classical problem-solving task.*

Key words: Problem posing, analogical transfer, cognitive framing, teacher training.

INTRODUCTION

The ability of transferring knowledge from a known, familiar context to a new, unfamiliar context is considered an important skill (e.g. Casale, Roeder, & Ashby, 2012). In the study of transfer, much of the research focused on the issue of analogical transfer in problem solving (e.g. Gick & Holyoak, 1983; Mayer & Wittrock, 1996; Bassok, 2003). Yet, more recently, there is an increasing interest for the study of analogical transfer in problem posing (e.g. Bernardo, 2001; Mestre, 2002; Kojima, Miwa, & Matsui, 2013).

In a previous paper, Singer and Voica (2017) studied the relationship between creativity and analogical transfer in a problem-posing situation. That study have used data from a sample of 97 high-achievers from grades 4 to 12 who responded to a problem-posing task; the study concluded that, in a population of high achievers, the use of far analogy allows for the differentiation of more mathematically creative persons.

To better understand the transfer mechanisms, we continued the previously mentioned study. In this paper, we report findings on a sample of university students who have been asked to generate new problems starting from an initially given problem (the same as in the previously mentioned research); as a difference, we asked the respondents to comment on the problems they posed from a (prospective) teacher's view. In this new research we were interested to find answers to the following question: In a problem-posing context, are pre-service teachers able to transfer more subtle elements of a problem, such as implicit solving indications contained in the problem text, or do they remain at the level of a superficial analogical transfer?

THEORETICAL BACKGROUND

Problem posing and analogical transfer

Analogical transfer supposes that, starting from a source context/domain, a new target context/domain is developed by mapping the components of the source onto it.

The study of analogical transfer in problem posing seems not to be very extensive. Among the examples we found, Bernardo (2001) describes a series of experiments for the study of "analogical problem construction", in which a group of high-school students posed word problems in basic probability. The conclusion of that study is that analogical problem construction contributes to the effective increase of the analogical transfer level. In a study conducted with high-performing university students, Mestre (2002) showed that, in physics, problem posing is a powerful tool for assessing the transferability of knowledge to new contexts. He found that students posed appropriate problems in many cases.

From another view, a large body of research has shown that transfer is difficult, but the ability to solve a problem naturally determines a near transfer, i.e. it allows solving a similar problem, in a similar context (e.g., Gick & Holyoak, 1980; Singer, 2008). In the present study, both the source context and the target context suppose: an input set; a rule of association; and an output set. In this case, near transfer means that the target context is built keeping the same nature of the source components, while far transfer supposes that, in the target context, at least one of the components is hardly/structurally modified.

Problem posing and creativity

There are various frameworks for studying creativity. A well-known model has been developed by Guilford (1967) and Torrance (1974); starting from this, mathematical creativity is currently described by three parameters: originality, fluency, and flexibility (e.g. Leikin, 2009).

In a problem-posing context, we found that a framework based on cognitive flexibility is very responsive to addressing questions related to students' creativity (Pelczer, Singer, & Voica, 2013; Singer & Voica, 2015; Voica & Singer, 2013). We consider that a student proves cognitive flexibility when (s)he generates new proposals that are (relatively) far from the starting item (i.e. cognitive novelty), poses different new problems starting from a given input (i.e. cognitive variety), and is able to change his/her mental frame in solving problems or identifying/discovering new ones (i.e. change in cognitive framing). We focus the present paper on this last characteristic of creativity, i.e. change in cognitive framing.

METHODOLOGY

During an in-service and a pre-service teacher-training program, participants have been challenged to pose problems starting from a given source.

A first project was implemented within a professional-conversion program for a group of in-service teachers who wanted to teach in primary education. Within this project, their assignment was that, starting from a given numerical context and an example of a problem, to pose as many problems as possible similar to the given one. Out of the 112 participants in the program, 71 students responded to this assignment. The preliminary results gathered from processing the data of this project have shown that the participants, although graduates from university, do not master analogical transfer at a basic level: they succeeded in posing new problems, but many of their proposals were actually not connected to the task. This striking conclusion made us to design a second project.

Within a mathematics-education course held under a pre-service teacher-training program, we asked students to respond to the following task:

> Start from the next multiple-choice problem:
>
> Consider the following associations: 36 → 18; 325 → 30; 45 → 20; 30 → 0. Find a rule that describes these associations and specify the number that matches 531 within this rule. Choose the correct answer from: A) 20; B) 40; C) 15; D) 31; E) 51.
>
> Solve the problem above, and then pose/invent 3 new similar problems starting from it. You have total freedom in proposing the new problem context, the association rules, the numbers in the statement or in the answer variants, the number of given hints, or in making other changes in the given problem text. (You can even personally interpret the meaning of the phrase "similar problems"!). Yet, formulate your proposals as multiple-choice problems with only one correct answer. For each new problem, indicate a solution and specify the details of the statement that (in your opinion) could drive the presumable solver to the correct solution.

Through this project, we sought to replicate and develop the topic addressed in a previous study mentioned above (Singer & Voica, 2017). More specifically, this time we were particularly interested in how the students of our sample assume the teacher's role and give guidance (implicit or explicit) to a possible solver for identifying the association rules from their posed problems. The project was completed by 15 students (out of a total of 42 students enrolled in the course). To search for an answer to the question that initiated this study, in the present paper, we use the data obtained from this sample.

RESULTS

The starting problem is open-ended: we can imagine different association rules between the given numbers. As a result, any rule found by the participants that could apply to all given associations, was *a priori* considered valid. However, by choosing input numbers and distracters, we were considering suggesting to the students the following association rule: a natural number corresponds to the product of its digits. For the present study, we found that the way the problem was formulated was fairly transparent because all the students who responded to the task identified and used the rule initially envisaged (i.e. a natural number corresponds to the product of its digits). This was not a surprise: in our previous research, which started from the same given problem, we found that all 97 respondents indicated this association rule (Singer & Voica, 2017).

The students in our sample have posed a total of 47 new problems (two of the students have posed 4 problems). At first analysis, we organized these problems into two categories, as we considered (as solvers) that the proposals offer or not some support points for the problem solution. The diagram in Figure 1 shows this classification.

For a more detailed discussion, we present in this section some relevant examples of the problems posed by students.

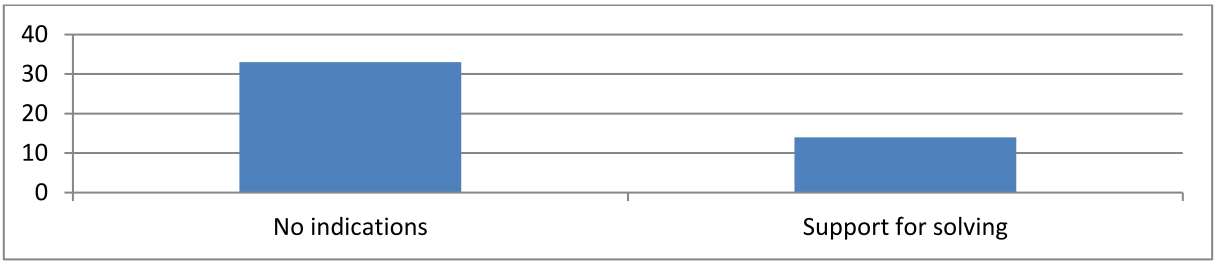

Fig. 1. The distribution of the posed problems according to the absence/presence of solving support

Example 1

Two of the problems proposed by Daniela are:

Problem 1. Consider the correspondences: 36 → 27; 325 → 15; 45 → 36; 30 → 9. Find the rule and state the number that matches 531:

A) 75 ; B) 522; C) 40; D) 25; E) 125.

Problem 2. Consider the correspondences: $f_1 = 2X^3 - 3X^2 + 1 \in$ □$[X] \to \{X - 1, X + \frac{1}{2}\}$;

$f_2 = X^4 - 4X^3 - 4X - 1 \in$ □$[X] \to \{X - i, X + i\}$; $f_3 = X^2 - 441 \in$ □$[X] \to \{X - 21, X + 21\}$.

The above associations are made following a certain rule. Under the above rule, circle the right answer for the following association: $f_4 = (X^2+1)(X^2+X+1) \in$ □$[X] \to$ A:

A) A = $\{X - i, X + i, X - (1 - i\sqrt{3})/2, X - (1 + i\sqrt{3})/2\}$; B) A = ∅(empty set);

C) A = $\{X^2 + 1, X^2 + X + 1\}$; D) A = $\{X - 1, X + 1, X + 2\}$.

In the first posed problem, Daniela indicated the association rule:

$\overline{xy} \to x \cdot (x + y)$; $\overline{xyz} \to x \cdot (x + y)$.

In the second problem, Daniela changed the context and moved to polynomials, to which she associated sets; more precisely, to a polynomial she attributed the set of irreducible and monic factors, defined over the commutative field given in the text.

Example 2

One of the problems posed by Ana is:

Problem 1. Consider the associations: 236 → 12; 325 → 1; 529 → 1; 830 → 24. These associations are made according to a certain rule. The number that corresponds to 438, following this rule, is: A) 32; B) 15; C) 20; D) 12; E) 4.

Starting from the initial problem, Ana invented a new rule of association: to an even number corresponds the product of the first two digits plus its last digit, and to an odd number corresponds the product of the first two digits minus its last digit.

Example 3

Two of the problems posed by Andra are:

Problem 1. Consider the number sequence: 0, 1, 4, 13, 32, 65 ... . This sequence continues following a certain rule. The number on the 8^{th} position according to this rule is:

A) 116; B) 115; C) 67; D) 189.

Problem 2. Consider the following associations:

4 → 2; 11 → 1; 22 → 2; 24 → 3; 15 → 3; 79 → 8.

> The associations above are made following a certain rule. The number that corresponds to 19 following this rule is:
>
> A) 5; B) 1; C) 3; D) 4; E) 8.

For problem 1, Andra indicated an association rule based on the observation of successive differences between the numbers of the sequence. Unlike other colleagues, she did a detailed discussion on how to tackle the problem, positioning herself in an obvious role of a teacher who is trying to guide her students into solving. For the other two problems, however, Andra's discussion is more restrained.

DISCUSSION AND CONCLUSIONS

The students of our sample were able to observe similarities and formulate consequences in order to identify an association rule for the given problem. They then succeeded to make a transfer and thus to manage posing new problems.

In some situations, the posed problems were consistently far from the given context, but in some others they just minimally deviated from it. An interesting case is, in this regard, Daniela's case. She uses near transfer to generate her first problem, keeping the same input numbers as in the given problem (including the number 531, referred in the given question) and only changed the association rule. But Daniela uses far transfer to generate the second problem, in which she associated polynomials with sets of irreducible factors.

As a strategy for generating new problems, students say they have done the following: first, they imagined an association rule, and then they made the problem statement. Given this succession, in which the solver's perspective is lost, the following questions naturally arise: Are the data the students provide in the problem wording enough coherent to suggest an association rule to a solver? Is there a single, common rule that related the data and might be revealed to a solver?

Some of the students from our sample insist on the need to keep a balance on the number of one-to-one associations we can include in the text of the problem, taking into account that, on the one hand, a small number of associations does not ensure the uniqueness of the rule and does not provide any clue to solve; on the other hand, too many one-to-one associations given as examples make the difficulty of the problem to decrease significantly.

However, as we can see in Figure 1, most students of our sample seem not being aware of aspects of a didactical nature. For example, Ana's problems do not provide enough clues for finding a plausible rule: she does not seem to be aware of the fact that she gives too few inputs for solving, and, consequently, her problems are practically inaccessible to a potential solver.

A different situation we encounter with Andra. She includes, in all her posed problems, a greater number of numerical associations, compared to the given problem (6, versus 4), and a variety of numbers. There is here a double aspect: on the one hand, Andra tries to preserve certain mathematical accuracy by carefully choosing numbers that lead to a unique rule; on the other hand, she pays attention to some didactical aspects – she provides more associations, willing to provide more support. She explains the following about the chosen numbers:

> "I chose both odd and even numbers, numbers with equal digits, numbers with distinct digits, numbers divisible by 2, divisible by 5, a prime number, I have chosen diversity."

Concerning cognitive framing, the students of our sample prove the understanding of the context of the initially given problem (natural numbers, generated on the basis of a consistently applied rule) and manage to pose new problems, some of them using surprising contexts (such as Daniela's proposal exemplified above).

Students design analogical problems by appropriately framing the new context. In some cases, they come up with adjustments that provide changes in cognitive framing. By imagining some (new) association rules or new contexts, it does not mean, however, that students transfer also more subtle elements of the initial problem to their new posed problems: one reason for that seems to be the fact that, in most cases, they do not seem to realize that a solver needs "clues" in the problem statement.

We interpret this behavior as proof of a superficial transfer, which only takes into account the mathematical aspects of the starting problem, and not its didactical aspects. In addition, students also seem not able to pay enough attention to a good definition of the problem conditions, a fact that deals with their mathematical competence. We thus noticed that, at least for our sample, students' level of mathematical competence was better made visible through this type of project activity than through a classical problem-solving task.

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PROBLEM SOLVING IN SECONDARY EDUCATION: A QUALITATIVE ANALYSIS OF THE DIFFERENCES BETWEEN HIGHLY AND MILDLY GIFTED STUDENTS

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Recent PISA studies revealed that students' problem-solving skills are in many countries not as well developed as their mathematical skills. Since highly gifted students are considered better problem solvers than mildly gifted students, the question is in which ways they behave differently. To answer this question, we analysed the problem-solving process of secondary school students at preparatory university level ($IQ \geq 110$) using an existing problem-solving model. Both junior ($N = 26$, grade 7-9) and senior ($N = 34$, grade 10-12) students make most of their errors in the analysis and verification phases. While the highly gifted students ($N = 5$, $IQ \geq 130$) encounter fewer problems in the analysis and verification phases, they make more sloppy mistakes in the planning and implementation phases. The findings from this study suggest that highly gifted students are indeed better at problem solving but could benefit from paying more attention to details and mathematical notation.

Key words: giftedness, problem solving, Mathematics, secondary education.

INTRODUCTION

Problem solving is a skill that is more and more in demand in modern society. However, in 2012, one of the conclusions from the international PISA studies was that in many countries, including the Netherlands, problem-solving skills are lagging behind compared to standard mathematical skills (OECD, 2013). Previous research on the problem-solving process mostly focuses on primary education whereas secondary education and the role of (highly) gifted students have not been fully explored yet (cf. Lesh, English, Riggs, & Sevis, 2013; Kolovou, 2011).

This paper describes an experiment in which 60 Dutch junior and senior secondary school students at preparatory university level ('VWO', $IQ \geq 110$) completed the same problem-solving test. Their answers as well as their problem-solving process have been analysed based on an existing problem-solving model (Rott, 2012). There were five highly gifted students within the group ($IQ \geq 130$). The same model has been employed to analyse and compare the processes observed in the results of these students to the processes from the other VWO students.

The research questions addressed are:

(1) Looking at Rott's model, at which stage of the problem-solving cycle do Dutch students pursuing preparatory university education ($IQ \geq 110$) encounter the most problems and what are the differences between junior and senior students?

(2) Looking at Rott's model, in which ways do highly gifted students ($IQ \geq 130$) perform differently than mildly gifted students, and what does this mean in relation to their problem-solving skills?

THEORETICAL BACKGROUND

What a problem is, is always relative to an individual (Schoenfeld, 1985). Problem-solving capabilities entail the ability to understand a problem, to devise a plan to solve the problem, to carry out this plan and to look back at the process to evaluate the result obtained (Pólya, 1990). Whereas early problem-solving models had a linear nature, the one that is currently used distinguishes the same phases, but views problem solving as a cyclical process (Rott, 2012; see Figure 1).

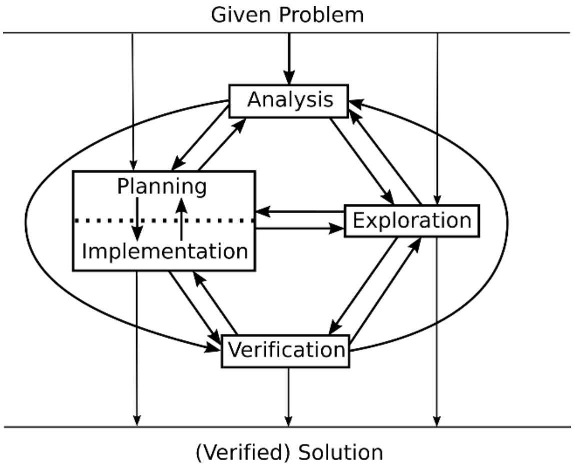

Figure 1: The problem-solving model proposed by Rott (2012)

Since problem solving is one of the 21st century skills (Ananiadou and Claro, 2009), it should be an important aspect of primary and secondary education, especially for gifted students who can achieve more than average students in this respect. Yet, previous research tells us that, in the Netherlands, the skills with relation to problem-solving capabilities are not as well taught as skills with relation to standard Mathematics (Doorman et al., 2007). In fact, the Dutch had the biggest difference in these two scores in the PISA-studies of 2003 (OECD, 2005). In the similar and more recent study, they still performed worse at problem solving than at standard Mathematics (OECD, 2013).

With regard to problem-solving skills in PISA studies, students were asked to draw conclusions, make decisions, troubleshoot and analyse the procedures and structures of a complex system (Lemke et al., 2003). Research related to this subject mostly focuses on primary education but lacks the elaboration of secondary education. Even the newest textbooks for primary education do not contain questions related to problem solving, and at best, this kind of questions can be found in extra materials for gifted, highly achieving children (Doorman et al., 2007). This means that these children, that already have an edge over their peers in this respect, are more exposed to problem-solving questions at a young age, which makes the gap between the two groups even bigger.

Studies on gifted students with respect to problem solving have revealed that they are faster than average-attaining students (Klein, 2017) and that they are precocious when it comes to the development of problem-solving skills (Threlfall & Hargreaves, 2008).

METHODS

In order to get more insights in the problem-solving process of highly and mildly gifted students throughout their secondary school period, a test has been composed consisting of problems that are not commonly found in Mathematics textbooks but could be solved by logical thinking and reasoning. The test contained several multiple-choice problems from the International Mathematical Kangaroo competition combined with several open-ended logical problems. The eleven problems increased in difficulty throughout the test. The last four problems were presented only to the senior students, as they required some basic mathematical knowledge (e.g. calculating angles). In this paper, the qualitative analysis focuses on the results of three of the open-ended questions: the signpost problem, the handshaking problem and the line drawing problem (see Figure 2).

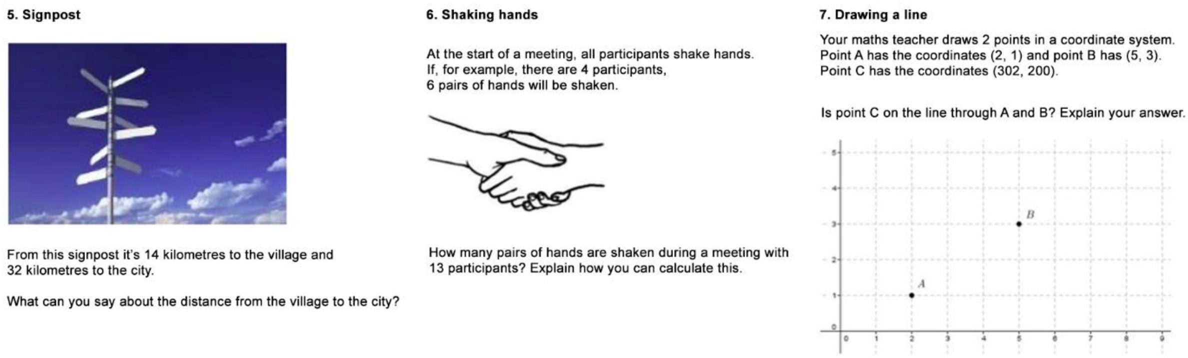

Figure 2: The problems analysed in this paper.

Since we are not primarily interested in the students' answers, but rather on the process that takes place behind obtaining an answer, the students not only answered the questions but were also asked to reflect on their problem-solving process through several questions that provided insights in the different phases from Rott's model. For all problems, the students were asked to indicate if they had seen this problem (or a similar one) before, and whether they knew how to begin solving the problem. For some of the problems, including problem 6 and 7, the participants were also asked to which extent they planned their solution before they implemented it, how organized they worked, how difficult they found the problem and in which ways they verified their answers (Schoenfeld, 1985).

To select participants, a computer program randomly selected twelve students for each grade (grade 7-12) of preparatory university education ('VWO'), the highest educational level at a Dutch secondary school. If more than four students from a certain grade did not participate, we randomly selected other students, from the same grade, as their replacements. In general, the majority of VWO students have an IQ that is higher than 110, which makes them all at least mildly gifted. Five students from our group of sixty students have been classified as highly gifted ($IQ \geq 130$) on the basis of an intelligence test that was taken outside school. The results from these five students have been compared to the results from the other students.

RESULTS

Quantitative analysis

The number of correct answers gives a first impression of the problem-solving capabilities. There is a significant difference in the scores of junior students ($N = 26, M = 3.19, SD = 1.50$) and senior students ($N = 34, M = 4.79, SD = 1.34$; $t(58) = -4.36, p < .001$). These results indicate that senior students score significantly better on the test than junior students. When looking at the results in more detail, the senior students outperform the junior students on each of the open questions 5, 6 and 7 (respectively $F(1, 58) = 6.45$, $p = .014$; $F(1, 58) = 7.40$, $p = .009$; $F(1, 58) = 5.70$, $p = .020$). Knowing the problem beforehand does not always lead to significantly better results. For the handshaking problem, the answer was significantly more often correct if the problem was familiar to the students ($\chi^2(1) = 8.30, p = .004$), whereas this was not the case for the signpost problem ($\chi^2(1) = 2.67, p = .103$) and the line drawing problem ($\chi^2(1) = 3.02, p = .082$).

As for the distinction between highly and mildly gifted students, we found that there was a significant difference in the Mathematics grades between highly gifted students ($N = 5$, $M = 8.04$, $SD = 1.14$) and mildly gifted students ($N = 55, M = 6.44$, $SD = 1.47$; $t(58) = -2.38, p = .021$). The highly gifted students scored thus significantly better on Mathematics than the other students. Also on the problem-solving test, comparing the highly gifted students to the mildly gifted students revealed that the gifted students ($N = 5$, $M = 5.60$, $SD = 1.67$) outperformed the mildly gifted students ($N = 55, M = 3.96$, $SD = 1.55$; $t(58) = -2.25$, $p = .029$). The highly gifted students thus scored significantly better on the test than the mildly gifted students. However, we must be very careful with our conclusions, since the number of students, and thus the effect sizes, are low.

Qualitative analysis

Phase one – Analysis

The analysis phase is about understanding the problem: does a student understand the question, and does he know what he needs to investigate? We observed differences between junior and senior students as well as differences between highly and mildly gifted students.

As for the older students, they could draw on more background knowledge, which made the analysis phase often easier. This was especially clear for the line drawing problem, which is similar to what students need to do when they work with linear formulas. Especially the older students explained they had seen it before in their Mathematics lessons. Making a function of a linear graph and checking whether a certain point is on that line is often practised during Mathematics lessons. However, this type of problem is only taught after 2 years of secondary school, so for the youngest students (grade 7-8) this was not a routine question. This meant that they did not recognise the problem, which forced them to put more effort into understanding the problem. In addition, older students read questions more carefully and are more used to questions in which more than one answer should be given. This could be seen in the signpost problem, where most junior students were satisfied with one answer while many of the senior students gave the complete solution.

In general, the highly gifted students recognised more problems and seemed to understand how to tackle new problems better than the other students. Even when the questions were completely new to them, they generally understood what was asked. Only for the signpost problem, two junior highly gifted students did not understand what they had to do to solve the problem. The way in which the question was formulated had likely confused them. One of them gave an answer and wrote it down in both metres and in centimetres, technically saying something about the distance.

Phase two – Exploration

In the exploration phase, students start looking for patterns and procedures to solve the problem. In order to understand what is happening, they need to critically analyse the information provided.

The importance of the exploration phase was most visible in the handshaking problem. What we noticed is that students looked at the information given (i.e. for four people there are six handshakes) and tried to come up with a method based on only this example. For example, one of the students concluded that $4 \cdot 1.5 = 6$, whereas they should have interpreted it as $3 + 2 + 1 = 6$ (or $\frac{4 \cdot 3}{2} = 6$), and therefore the answer was incorrect. Such mistakes are the consequence of not fully exploring a problem but being satisfied after trying one approach. All of the highly gifted students saw the underlying pattern and solved the problem correctly. One of the senior students even used a completely different method and solved the problem in a unique way.

Looking at the line drawing problem, many junior students who did not know the problem did not even try to solve it, while some others, including the highly gifted students, applied their knowledge of ratios. This is a common strategy in the exploration phase: comparing a new problem to known problems. They compared (302, 200) to (3, 2) and took this as their starting point. Some of the students who should recognize this problem from Mathematics class (grade 9-12) did not see the similarity, even though this is a very common question. All highly gifted students took the mathematical approach and calculated the slope of the line.

To sum up, what we noticed was that most students who did not understand a problem just guessed instead of trying to figure out new ways to tackle it. In contrast, none of the highly gifted students guessed any answers; they all tried at least one thing.

Phase three – Planning and implementation

After a problem has been analysed and explored, the next step is finding the solution. Since we were interested primarily in the process instead of getting the correct answers, sloppy mistakes were not counted. For example, one student understood the handshaking problem, but made a small mistake in adding up all the numbers. All three highly gifted junior students wrote down the explanation for the handshaking problem but forgot to write down the actual answer. The problem asked students to explain how to answer the problem, but they also had to answer how many pairs of hands would be shaken. Since they clearly did see the pattern, we marked their answers as correct.

Phase four – Verification

After a problem has been solved, the solution should be verified: is the solution correct and is it complete? Students generally did not check their solutions and, as a consequence, they did not notice that some answers were impossible. For example, students who came up with 19.5 handshakes did not notice that this is impossible. One highly gifted student used the example to check his method. Another junior student indicated that she did not check her answer, because she did not understand how to tackle the problem. When a student got stuck during the analysis phase, they could not make and carry out a plan or verify it. Students who did check their answers said they just carried out their plan a second time instead of trying to find another approach. Even the highly gifted students who checked their answers just calculated the same thing again.

CONCLUSIONS AND DISCUSSION

Mathematics books for secondary school students focus on developing mathematical routines instead of learning how to think critically. The results from this study confirm that students indeed are better at implementing solutions to known problems, while they struggle with non-routine problems. Both junior and senior students encountered most problems in the analysis and verification phases. They often did not read a question carefully and they generally did not look back to check if they answered the question correctly. Secondary schools should think about the way in which they currently teach Mathematics and consider paying more attention to the 21st century skill of 'problem solving' instead of focusing mainly on the implementation phase.

Highly gifted students outperform the mildly gifted students when it comes to identifying patterns in non-routine problems. However, they are worse at executing their plans and writing down the process. Highly gifted students made more errors in the implementation step, whereas the other students made more errors in analysing the problems. The highly gifted students could profit from learning more about the problem-solving process and how to make less unnecessary mistakes in their work. It is notable that these students have significantly better Mathematics grades than the other students, which might suggest that they do not see the need to change their approach. They compensate for their sloppy mistakes by their better understanding of the topics. It may be a good idea to change part of the education system to better fit the special needs of highly gifted students. They need training just like the other students, just on different subjects, such as being more precise and making more effort in writing down the answers in detail.

It would be interesting to follow up this research with a second bigger one, testing a different school or generation of students to get more accurate results. Another relevant question is how the educational system could be changed to increase students' problem-solving skills and to better accommodate the needs of highly gifted students.

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PROSPECTIVE TEACHERS' VIEWS ON MATHEMATICAL GIFTEDNESS AND ON TEACHERS OF MATHEMATICALLY GIFTED STUDENTS

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Abstract. *The aim of the study is to examine prospective secondary mathematics teachers' views of mathematical giftedness with respect to characteristics of the teachers of the mathematically gifted students. The data were collected through a focus group interview with seven prospective secondary mathematics teachers. Then, the data were analysed by using descriptive analysis. The participants associated mathematical giftedness with genetic factors, social environment, and effort. It is concluded that the participants' views on mathematically gifted students are limited to their personal experiences with such students and their responses were seen to be under the influence of related films and books they have encountered. They also defined the teachers of mathematically gifted students with regard to personal and professional characteristics.*

Key words: Mathematically gifted students, secondary level, prospective mathematics teachers, teacher education

INTRODUCTION

Although giftedness draws considerable interest in mathematics education literature, there are no clear definitions of mathematical giftedness (Karp, 2009; Wieczerkowski, Cropley, & Prado, 2000). Instead, the characteristics of mathematically gifted students have been described in the recent literature (Singer, Sheffield, & Leikin, 2017; Sriraman, Haavold, & Lee, 2013). Krutetskii's (1976) longitudinal work is one of the outstanding studies, which offer a wide range of descriptions in this area. There is an approach in the 1990s that involves a broader group of students which defines mathematically promising students (Sheffield et al. 1999). This approach focuses on students' mathematical potential that is a function of variables such as ability, experience, and motivation that could be maximized when realized in learning settings (Leikin, 2011; Sheffield et al. 1999). In the classrooms, teachers are one of the agents who are responsible for differentiating the curriculum to support the gifted students and implement activities to nurture students' potential (Leikin, 2011). They can create learning opportunities to foster and realize students' potential and they should facilitate learning for the gifted students (Krutetskii, 1976; Milgram & Hong, 2009; Rowley, 2008). In that regard, the teacher has an important role in organizing a learning environment for gifted students.

Literature gives considerable insight for the characteristics of a good mathematics teacher (e.g. Wilson, Cooney, & Stinson, 2011). However, despite the importance of teacher's role in nurturing gifted students, studies on the characteristics of the teachers of the mathematically gifted students are limited in number (Karp, 2010; Karsenty, 2014; Leikin, 2011). In a regular classroom, teachers should also know how to differentiate the students who have mathematical potential and characteristics of the mathematically gifted students (Miller, 1990). This study aims to reveal readiness of the prospective mathematics teachers who will also be responsible for the education of gifted students in their classrooms. Therefore, this study investigates prospective secondary mathematics teachers' views of mathematical giftedness and the characteristics of the teachers of the mathematically gifted students.

THEORETICAL FRAMEWORK

Krutetskii (1976) revealed a number of characteristics related to obtaining mathematical information, processing mathematical information, and retaining mathematical information. The research found that teachers have difficulties while evaluating mathematical giftedness and they mostly tend to focus on computational skills and school success of their students as a talent evidence (Koshy, Ernest, & Casey, 2009; Miller, 1990). The literature exploring the characteristics of mathematically gifted students is vast, but the studies devoted to the characteristics of mathematically gifted students' teachers are few in number. For instance, Leikin (2011) investigated the profile of a teacher who have experience in teaching mathematically gifted students for years in Russia. She revealed that the teacher should have strong subject matter knowledge in mathematics. Also, the teacher should be able to differentiate the abilities of students individually and challenge each student according to these abilities in the classroom setting. Being supportive and respectful for the students are both important characteristics of the teacher of the mathematically gifted students. The teacher should have enthusiasm towards mathematics and a sense of humor for creating positive atmosphere and joyful learning. Leikin (2011, p. 86) concludes that "the expert teacher of gifted students addresses in his teaching all the components of students' mathematical potential (intellectual, affective, and social) and in this way provides students with multiple opportunities for the realization of their potential." Karsenty (2014) also highlights that the teachers of mathematically gifted students should have strong mathematical knowledge, pedagogical skills, and personal-social qualities especially at the secondary level. The remarkable conclusion of the study was that the interactions with gifted students improved the teachers' both subject matter and pedagogical knowledge. Similarly, Karp (2010) revealed that working with gifted students have an influence on teachers' personal improvement and also concluded that developing the student's mathematical creativity required the teachers' own creativity.

METHODOLOGY

The aim of the study was to investigate prospective secondary mathematics teachers' views on mathematical giftedness and on the characteristics of the teachers of the mathematically gifted students. Participants were selected from prospective teachers who were senior students in the secondary mathematics teaching program of a state university in Turkey. The participants mostly took all of the courses that include mathematics and pedagogical courses.

In this study, the data were collected though a focus group interview with seven prospective mathematics teachers named as Ayşe, Kumru, Seval, Onur, Gamze, Rana, and Nida (pseudonyms). Five open ended questions were asked to the participants (Table 1). The focus group interview lasted for approximately 90 minutes and the interview was recorded by a video camera. Related prompts were employed during the focus group interview in order to stay focused on these five open ended questions. The collected data were coded by using descriptive analysis.

1. What do you understand from the term “giftedness”?
2. What do you understand from the term “mathematical giftedness”?
3. What do you know about the mathematically gifted students? Do you ever have any experience with such students?
4. What could be the characteristics of the mathematically gifted students for you?
5. What could be the characteristics of the teachers of the mathematically gifted students for you?

Table 1: Focus Group Interview Questions

FINDINGS

Prospective Teachers’ Views on Mathematical Giftedness

This subsection focuses on the prospective teachers’ views on mathematical giftedness, which they related with various concepts such as genetic factors, social environment, effort, and IQ scores. It also addresses some of participants’ views about mathematically gifted students.

The prospective teachers mostly related mathematical giftedness with genetic factors, social environment, and effort. Rana thought that genetics determines the boundaries for the giftedness and she added that “[…] absolutely environment is an important factor; social environment can bring the giftedness to a higher level but not to a lower one”. All the prospective teachers, except Ayşe, shared the same thought: people can improve their giftedness, but it is not possible to lose their giftedness. However, only Ayşe explained her ideas as following:

> It seems to me that you cannot stay at the same level without doing anything. No matter how gifted you are, if you do not do something or do not go over something you have, you cannot improve your giftedness. You cannot just stand still; I think the brain begins to forget. No matter how gifted you are, you have a memory in the end.

As seen in her statement above, Ayşe considers it possible to lose someone’s giftedness without working on it. On the other side, the prospective teachers have different perspectives on connections between giftedness and effort. For example, Kumru said that gifted students make an effort while learning something and they are able to succeed with this effort. Most of the other prospective teachers did not agree with Kumru’s thought. Ayşe, one of those who followed this argument, explained this by the following statements: “If the gifted students have interest in something then they make an effort. The fact that they are gifted does not necessarily mean that they will make an effort or deal with anything.”

Some of the prospective teachers related giftedness with IQ scores. When the researcher asked them to describe what they know about IQ or IQ scores, they only described the relation between the scores and giftedness, not the structure and content of the IQ tests. Seval stated her opinion as follows: “I only encountered with IQ while I was reading some text on the internet.” She added that she related IQ and giftedness because of this reading.

The prospective teachers mostly described the mathematically gifted students according to their limited experiences with these students and the influences of films and books. For example, Seval and Onur said that they attended “Bahar Mathematics Meeting” in the autumn of 2018 that offered a series of seminars addressing everyone who are interested in mathematics. When they attended the seminar, which aimed to explain the graph theory, they came across with a student at a secondary level who attended the seminar actively. Onur stated that this student corrected something related to graph theory on the board which he might had never seen before. Seval said, "He also associated a concept in the graph theory with another concept which I cannot remember now." Gamze, who mentioned that she has never had an experience with a gifted student, claimed that gifted students have antisocial personality, which must have been a claim derived out of the influence of a film about a gifted person that she watched before.

The prospective teachers gave some details about the characteristics of the mathematically gifted students. For instance, Nida added that the mathematically gifted students think differently in learning the details of a mathematical concept from other students. She highlighted that "the gifted students want to investigate and rationalize a mathematical subject along with the reasons in detail […]" Similarly, other participants stated that mathematically gifted students do mathematical operations very fast, have pleasure while doing mathematics, and prefer to engage with complex constructs.

Prospective Teachers' Views on Characteristics of the Teachers of the Mathematically Gifted Students

The prospective teachers described the characteristics of the teachers of the mathematically gifted students in terms of teachers' personality traits, having advance subject matter knowledge and pedagogical advance (content) knowledge, following the new research in the literature, applying the special intuitions related to gifted students. The secondary mathematics prospective teachers emphasized the personality traits as one of the characteristics of the teachers of the mathematically gifted students. One of the dialogues among the prospective teachers explains this view as follows:

> Gamze: Sympathetic, progressive, enterprising
> Rana: Sympathetic, absolutely open-minded
> Onur: If not open-minded, there would be ego issues
> Ayşe: The teacher should not have a big ego

Kumru stated that a teacher of a mathematically gifted student should have in-depth knowledge of the mathematical concepts and should also go beyond the content in the curriculum. Her explanation is as follows:

> I think the teachers should improve themselves. For example, mathematics is our field and I have to remember all the concepts that we have discussed in Introduction to Algebra course and I should be able to teach them even 10 years after my graduation from the university […] I need to acquire even more advanced knowledge in Algebra.

Kumru explained that teachers should keep their subject matter knowledge active as if they were university students. She added that the knowledge should be also advanced in comparison to the secondary mathematics level. All the other participants agreed with her thoughts. In a similar manner, Rana continued with her thoughts related to both pedagogical and pedagogical content knowledge:

> It is not enough to know; it is also important to know how to teach the subject. Yes, we can have enough knowledge to answer the gifted students' questions, but how do I teach or show the right way? Of course, pedagogical formation is essential: we learn how to instruct mathematics in the university. But, the teacher of a mathematically gifted student should know how to instruct that student, it is very important, I think.

As stated above, Rana explained that it is not only enough to know the subject matter, it is also important to know how to share this knowledge with gifted students. She also stated that as prospective teachers, they have really little knowledge about how to teach mathematically gifted students. Here she mentioned that it is essential to have advanced pedagogical (content) knowledge for the teachers of the mathematically gifted students.

The prospective teachers proposed that a teacher of the gifted students should apply to special institutions such as counselling centres or schools for gifted students, if any in order to get support. Here is Rana's explanation: "[…] if there are such institutions, they will be very useful to have a face to face contact in terms of improving our knowledge on the gifted students". Kumru described the ideal teacher of mathematically gifted students as the one who follows the latest research in the literature and attends the conferences:

> If I were a teacher of a mathematically gifted student, I would not only improve myself in mathematics but also in the educational sciences. I mean I would follow some events like conferences and seminars. If I had a gifted student, I would read articles related to these students and develop a new educational perspective according to these experiences and readings.

As seen in her statement above, Kumru believes that searching and reading articles related to gifted students could help the teachers who have mathematically gifted students in their classrooms. Considering the thoughts of the prospective teachers, it is possible to say that they describe the teachers of the mathematically gifted students with regard to both personal and professional characteristics.

DISCUSSION

This study revealed that the prospective secondary mathematics teachers had limited knowledge on mathematical giftedness. They mostly stated their thoughts about mathematically gifted students according to computational skills of the students (Koshy, et al., 2009). It is determined that the participants insisted on some myths related to genetics, social skills, and developing talent (Sheffield, 2017). This may be due to not having much experience with gifted students. When the participants had some experiences with gifted students such as Seval and Onur, they were able to give examples that are more concrete.

According to findings on the characteristics of the teacher of the mathematically gifted students, it is possible to say that they firstly emphasized the importance of advanced knowledge in both mathematical subject matter and pedagogical approaches as stated in the literature (Karsenty 2014; Leikin, 2011). They also emphasized that following the latest literature would improve the skills of the gifted students' teachers. On the other hand, they mentioned the significance of the personal traits of the gifted students' teachers. Consequently, the participants were not able to give more detailed information on mathematical giftedness and on the characteristics teachers of mathematically gifted students. The reason for this is due to the fact that they did not gain any professional experience about mathematical giftedness throughout their teacher education. As Karp (2010) stated, sharing the experiences of the effective gifted students' teachers or working with gifted students could help to improve prospective teachers' skills. Thus, it is possible to recommend the inclusion of special education institutions where prospective teacher get the chance to interact with gifted students into the teaching practice course.

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IMPROVING MATHEMATICAL CREATIVITY IN THE CLASSROOM: A CASE STUDY OF A FOURTH-GRADE TEACHER

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Abstract*. This study explores a fourth-grade teacher's conceptions about mathematical creativity and their development after his attendance of a program concerning the promotion of creativity in mathematics teaching. The teacher completed one questionnaire before and one after the program whereby he was asked to choose a mathematical task that he considers suitable for promoting creativity and to explain the reasons for choosing it. Interviews were also conducted, before and after the program, to highlight his envision of creativity in mathematics and the impact the program had on him. Findings revealed enrichment of his conceptions after having attended the program, mainly with regards to generating and using original tasks as well as to the ways he deals with mathematics teaching for creativity.*

Key words: mathematical creativity, primary school teachers, training program

INTRODUCTION

For a long time, mathematical creativity has been related to the original work of mathematicians with great emphasis mainly put on algorithmic approaches to mathematics. However, creativity in mathematics has recently been linked to the ability to produce something new (Bolden, Harries, & Newton, 2010), even when the product is already known to others (Sriraman, Yaftian, & Lee, 2011). It is associated with employing divergent thinking and the ability to produce many solutions to open–ended problems (Kwon, Park & Park, 2006) as well as with non-algorithmic decision making (Ervynck, 1991). Creativity in mathematics is often assessed on the basis of the four indices of creativity proposed by Torrance (as cited in Silver, 1997). These involve fluency (the ability to produce many solutions to a given task), flexibility (the ability to employ different strategies in order to solve a task), originality (the ability to find new and statistically infrequent solutions) and elaboration (the ability to incorporate detail into the solutions).

Given that mathematical creativity is recognized as a dynamic characteristic of our mind (Leikin, 2009), the importance of fostering it in educational settings has been well acknowledged in recent education reforms proposed in mathematics curricula in many countries. However, it seems that children's potential for creativity can either be fostered or inhibited, depending on both the choices teachers make in terms of tasks that could develop children's creativity in mathematics (Levenson, 2015) as well as the teachers' ability to transfer the pedagogical knowledge for mathematical creativity into their mathematical teaching practice (Panaoura, & Panaoura, 2014). For example, Kattou, Kontoyianni and Christou (2009) found that although primary school teachers acknowledge their role as professionals for fostering mathematical creativity, they report factors of the educational system (e.g., restricted time, textbooks) that they themselves consider as barriers toward mathematical creativity excluding themselves from accepting any responsibility. Similarly, in-service primary school teachers in Greece seem to relate mathematical creativity to book-guided problem posing and mainly focus on algorithmic calculations, indicating their narrow conceptions regarding creativity in mathematics, whereas prospective teachers put emphasis on arousing student interest (Desli, & Zioga, 2015). Moreover, pre-service teachers in UK were found to mostly associate creativity in mathematics with the use of resources and technology and therefore their intention is to create a "fun" environment in the classroom (Bolden, Harries, & Newton, 2010). Teachers' intentions to 'teach creatively' rather than 'teach for creativity' can be seen in the abovementioned studies among others.

Training teachers in creativity development might ensure the success of teaching for creativity in mathematics. To this direction, Levenson (2015) and Shriki (2010) organized training courses in order to support teachers in terms of mathematical creativity. Their outcomes show that teachers found these courses helpful as they changed their perspectives and got them prepared to choose tasks that will enhance their students' mathematical creativity. However, as Levenson (2015) points out, we cannot be sure whether the skills acquired will be employed in the classrooms. Considering Mann's suggestion (2006) that emphasis on creativity is placed on creating authentic learning situations which allow students to think, feel and solve real problems, it is important to clarify how the idea of mathematical creativity is implemented and referred to in a mathematics lesson.

The present study constitutes part of a larger research project, which attempts to research this question in depth. It does so, by gathering information about seven in-service teachers' conceptions and teaching practices via questionnaire, interview and observation of mathematics lessons, before and after their attendance of a training program concerning mathematical creativity. In the present study the focus is narrowed down to an in-service primary school teacher's conceptions of creativity in mathematics and his ability to promote creativity after the attendance of a program on enhancing mathematical creativity in the mathematics classroom. Specifically, this study attempts to answer the following questions: a) How does the teacher envision mathematical creativity? b) Which task characteristics does he relate to mathematical creativity? c) What changes occur (if any) in his conceptions after his attendance of the training program?

METHOD

Similarly to previous case studies that attempt to identify changes in teachers' conceptions, (e.g. Levenson, 2015), the present study refers to a case study in which the methodology of collecting information at different times and by different means is applied. Although it is known that case studies provide data that are not enough to make general conclusions, they give us the opportunity to study changes in a person's conceptions, that may be subtle but of great importance. As Cohen, Manion and Morrison (2007) point out, case studies are not ideal for generalization, but they can portray 'what it is like to be' (p. 254) in a particular situation (e.g., in a classroom). Thus, they allow us to look at a case in its real life context -in our case, how teachers react to real situations- enabling readers to understand other similar situations.

Participant. Jason was an in-service 4th grade teacher with 18 years of teaching experience. Parallel to school, he is a PhD student in mathematics education. He was selected purposefully from among other teachers, who voluntarily attended a program aiming to present, employ, assess and discuss ways of fostering creativity in mathematics, based on his mathematical interest and willingness to participate in the study.

The program. The training program was conducted with the involvement of the first author and lasted for a total of 12 hours, equally distributed in four sessions. It provided the participants with the opportunity to analyze aspects of creativity in mathematics both theoretically by discussing research findings related to mathematical creativity and looking at the ways of promoting it and empirically by applying these when developing lesson plans for teaching mathematical concepts. The participants were encouraged to work cooperatively in order to modify mathematical tasks used within the program.

Instrument. Data were collected via a questionnaire and a semi-structured interview. The questionnaire asked Jason to initially choose a task that he considers appropriate for fostering his students' mathematical creativity, state the source of it and then explain the reasons for choosing the particular task. Interviews were also conducted in order to explore a deeper understanding of Jason's envision of creativity in mathematics with questions referring to the characteristics of a creative student and a creative teacher in mathematics as well as the appropriate environment for developing students' creativity with the use of creative tasks. Although supplementary data on Jason's teaching mathematics for creativity were additionally gathered via observation, their analysis will not be described as it goes beyond the purpose of the present paper.

Procedure. In order to evaluate the impact of the program on his conceptions about mathematical creativity, Jason was asked to fill in the same questionnaire twice: one month before the beginning of the program and two months after the completion of the program. The same procedure was followed for the interviews that lasted approximately 20 minutes each and were recorded.

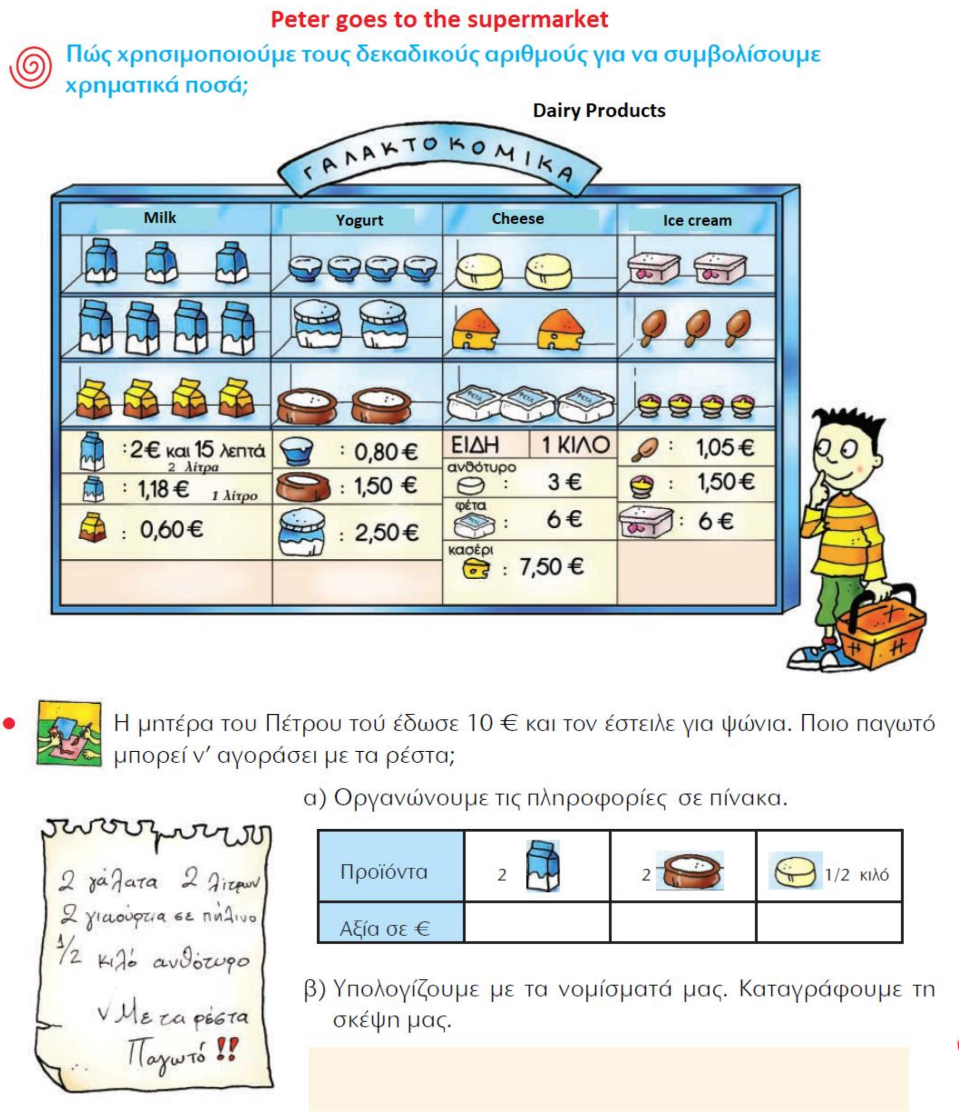
RESULTS

Jason's mathematical creativity before the program

Jason chose the task with the title 'Peter goes to the supermarket' shown in Figure 1 which is taken from the Grade 4 school mathematics student book. According to him, '*This task doesn't have a single solution. It enables students to make different combinations according to their own preferences and interests*'.

He seems to relate mathematical creativity with the possibility of getting many answers to a task (fluency) and with students' interests. However, the task he has chosen has actually only one solution, since mother's note gives specific instructions to Peter to buy particular dairy products, the cost of which is presented in the table that accompanies the task. The task needs to be modified in order to allow students with more solutions. Thus, although Jason identified multiple solution methods as a characteristic of mathematical creativity, his choice of task did not fit well with it.

Supplementary data from Jason's interview before the program show that he sees mathematical creativity connected to a solid mathematical background which can be useful for everyday problem solving: '*Mathematical creativity is taking advantage of our mathematical thinking and background in order to solve our everyday problems… Every person can be mathematically creative. However, advanced mathematical thinking and skills can be really useful… The main goal of mathematics teaching should not be algorithmic thinking, but finding creative ways to solve real-life problems*'. Taking advantage of students' everyday life events and personal interests and using real situations for students to apply their knowledge are of the utmost importance for Jason, concerning creativity: '*I think that every authentic problematic situation can provide us with the chance* *to develop creativity… Something from students' lives, something that the children have experienced with'*.



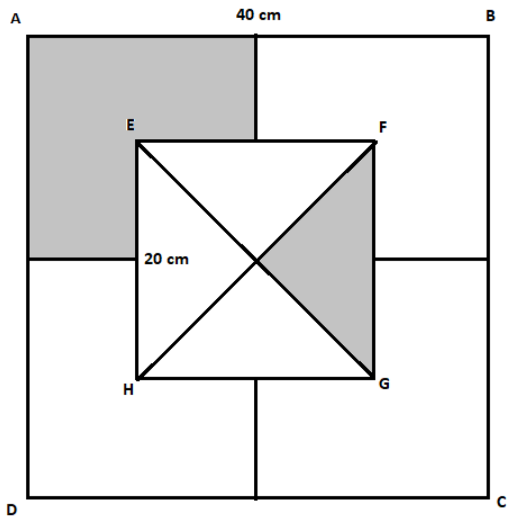
Peter's mother gave him 10€ for shopping. Which ice cream can he buy with the change? a) We organize the data in a table. b) We calculate using our coins and write down our thinking.

Mother's handwritten note in the left side says: "2 cartons of milk of 2lt, 2 clay bowls of yogurt, ½ kg cream cheese. Keep the change for ice cream!!"

Figure 1: Jason's task before the program

Jason's mathematical creativity after the program

Jason proposed a task (Figure 2) which he had handed in to his students a few days prior to the completion of the questionnaire. The task was his own devising and was inspired by tangram activities that the classroom had recently experienced with. He considers it a creative one: '*Students can solve this task by using many different strategies and employing their imagination. Although it has a single solution, it is open to many solving approaches. Moreover, not only requires students to recall to their previous knowledge about square's properties but also to use this knowledge when making combinations with different shapes*'.



ABCD and EFGH are squares. Calculate the shaded areas.

Figure 2: Jason's task after the program

This task does indeed favor the promotion of creativity, as many strategies (fluency) of different kind (flexibility) can be used and cultivates students' imagination. It is a non-routine task -quite different from the more conventional tasks found in the standard classroom textbooks- that requires students to think in a non-algorithmic way and break away from stereotypes. It allows students both to recall their previous knowledge on the properties of squares and explore it in order to find ways of calculating the triangle's area, although its formula has not yet been taught. It enables for different strategies to be discussed in the classroom and original solutions to emerge (originality).

Constant to his first interview, Jason still relates mathematical creativity to deep mathematical thinking and to solving everyday life problems. In this sense, '*Mathematical creativity is mainly associated with the development of mathematical thinking – one needs to have a background in order to move forward- and the ability to use this knowledge in order to solve real life problems*'. However, his conceptions of tasks that develop creativity have been quite enriched as he is found to look for and attribute more traits of creativity in his choice of task. '*Creative mathematical tasks can be open-ended tasks which do not have a single solution; tasks that enable students to employ many different strategies in order to be solved; situations that do not seem to be mathematical ones, but could be solved using mathematical strategies'.* He also suggested teachers should provide opportunities for students to form connections between new information and previous knowledge and experiences. He particularly made reference to the *"mathematical armory"* on which students rely and use in order to solve problem as an index of mathematical creativity. Interestingly, he raises the issue of developing a learning environment in which democracy is practiced: '*Democracy in the classroom is essential in order to promote creativity. Students should feel free to express their opinions without the fear of being criticized… All ideas are worthy of discussion*'.

The changes in Jason's conceptions about mathematical creativity are summarized in Table 1.

Creativity is related to:	Before the course	After the course
Open ended problems/ many solutions	✓	✓
Novelty	✓	✓
Connection to students' everyday life/ interests	✓	✓
Mathematical thinking/ background	✓	✓
Non-algorithmic thinking	✓	✓
Safe and democratic environment, freedom of speech	✓	✓
Many solution strategies	-	✓
Connecting new concepts with previous knowledge	-	✓
Imagination	-	✓

Table 1: Jason's perceptions of creativity that occur in his interviews and questionnaires

DISCUSSION

From the analysis presented in the previous section it is revealed that prior to the program Jason was aware of the major research findings related to teaching mathematics and its connection to real life problems. However, his envision of mathematical creativity has been enriched after the program, mainly with regards to generating and using original tasks as well as to the ways he deals with mathematics teaching for creativity. He now seems to be more confident to relate mathematical creativity to different solution strategies, to the use of imagination and to the connection of new concepts with previous knowledge. His orientation is clearly towards *what* he teaches, identifying the elements of creative mathematical tasks. He also endorses explicitly criteria that are compatible with his own professional values for *how* he teaches. His strong mathematical background and interest could be seen as contributing to preserving his initial conceptions and directing him in placing emphasis on mathematics teaching for creativity. However, it is interesting to further investigate the impact of such a training program on teachers who are not considered as experienced and professional active ones as Jason.

Although data from the present study are quite limited to providing information from only one in-service teacher, it still becomes evident that training programs related to mathematical creativity can be very helpful (Levenson, 2015), even for teachers who are already familiar with the recent research trends on mathematics teaching. Teachers' understanding of creativity in mathematics may be fostered if they are further trained to the pedagogies of creativity, that is to the ways they can develop their students' creativity.

It is essential that in order to develop learning environments that will boost students' mathematical creativity, the factors that underlie creativity on which education could have an influence need to be identified. Future studies could further investigate it and, thus, teacher preparation can be further effectively supported.

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PROPOSALS FOR PRACTICE

MAGIC POLYGONS AND ITS USAGE IN WORK WITH GIFTED PUPILS

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Abstract. *One of the gifted, mathematically promising pupils' needs is to work with recreational, challenging tasks. Very rarely, mathematics teachers themselves create and work with such tasks, so it is useful to encourage them and introduce them to new topics. Magic polygons are little known but very suitable in work with gifted pupils. Such polygons can also be a good source for problems in mathematics Olympiads, competitions, and for pupils' and students' research work. This article provides both simple introductory tasks as well as difficult and unsolved mathematical problems.*

Key words: Golygon, magic polygon, polyomino, polyiamond, perfect polygon, tiling.

INTRODUCTION

Mathematically capable (talented, gifted) pupils need tasks that go far beyond typical examples, exercises, and boring tasks from school textbooks. In other words they need to have enough complicated, challenging mathematical problems that invite them as well as their teachers to raise questions, to make conjectures, to create new definitions, to argue in order to explain, clarify, and revise their mathematical ideas and problem-solving processes. It is important that teachers' mathematical vision is not narrow, that it also includes some unexplored or little-learned mathematical islets, in research of which a gifted pupil can make some contribution.

Magic polygons is one of such topics; it is very good in developing different and innovative approaches by inviting teachers and pupils together to apply new communication methods in the learning of mathematics that could be fun and enjoyable at the same time. The mathematics should be organized around the so-called big ideas (extremum, invariant, one-to-one correspondence, symmetry, transformations, etc.) (Springer, 2014).

Here by *a magic polygon* we mean a squared or triangular polygon (a polyomino or polyiamond respectively) with all distinct whole sides: 1, 2, up to *n*. If side lengths of a magic polygon are in the increasing order, it is called *perfect*. In literature (Dewdney, 1990; Sallows et. al., 1991) perfect polygons on a square grid are also known as *golygons* – polygons containing only right angles, such that adjacent sides exhibit consecutive integer lengths. "The path starts at lattice line with a segment of unit length, turns 90 degrees in either direction, continues for 2 units, turns again in either direction, continues for 3 units, and so on."

When starting work with pupils, a good way to introduce magic polygons is to prepare and [to] hand out the square and triangular grid worksheets. Then the next step would be to ask do some simple (warm up) tasks. For example, "Draw a path on a given triangle grid how a grasshopper can jump if it can make 6 jumps, each 1, 2, 3, 4, 5, 6 units long and same length jump it can make only once. At the end he has to get back to the place where it started his first jump." Can such a path be drawn on a square grid? If not, why? Moreover, what happens when the grasshopper can make 8 jumps, each 1, 2, 3, 4, 5, 6, 7, 8 units long? Can pupils find such paths on both grids – squared and triangle? It is not obvious that such path exists. However, with a little experimentation, the pupils will find some of the required polygons. When finding of magic polygons has been mastered at the first level, we can raise a number of the next level tasks, etc. How many cells (squares or triangles) have your polygons? Can you get a bigger area than the ones you found? Which type (see Fig. 1) of octagons did you find?

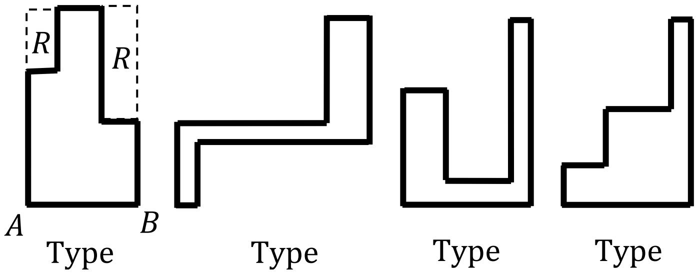

Figure 1. Octagons on the square grid

Did you notice that a perimeter of your polygon on the square grid is even number? Why is that so? Now look at the 6-gons you found on the triangle grid. Why does all of your 6-gons have odd area? Problem "Does there exist a hexagon (not necessarily convex) with side lengths 1, 2, 3, 4, 5, 6 (not necessarily in this order) that can be tiled with a) 31 b) 32 equilateral triangles with side length 1?" was proposed by A. Cibulis for Baltic Way competition in 2016. Latvian team leader Māris Valdats (Baltic Way solutions, 2016) solved this problem in the elegant way.

Recall that a polyomino (a polyiamond) is a plane figure formed by joining unit squares (unit regular triangles) edge to edge. The classic reference book on polyominoes is (Golomb, 1994).

Problems such as necessary and sufficient conditions of existence of magic polygons, their possible areas and some others have been proposed by A. Cibulis, see (Cibulis, 2016) as well as the master theses (Bulina, 2018). (Cibulis, 2016) also briefly discusses Latvian and Lithuanian pupils' findings on magic polygons.

It is not difficult to prove that the smallest magic polygon is an octagon on the square grid (Fig. 2), and a pentagon on the triangle grid (see Fig. 3). The necessary and sufficient condition of existence of magic polyominoes (n-gons) is $n = 4k$ while necessary and sufficient condition of existence of perfect polyominoes is $n = 8k$. This result and the fact that every polyomino can be transformed in a polyiamond immediately implies the existence of magic polyiamonds for $n = 4k$, and existence of perfect polyiamonds for $n = 8k$. With a simple construction, one can prove that a magic polyiamond n-gon exists for every $n \geq 5$.

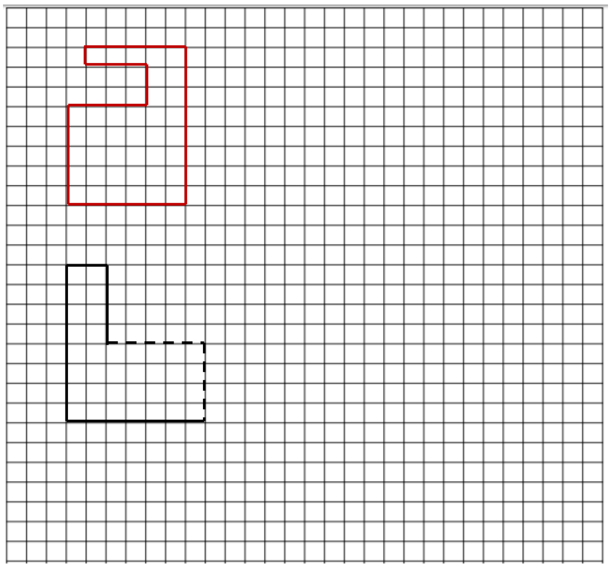

Figure 2. The square grid

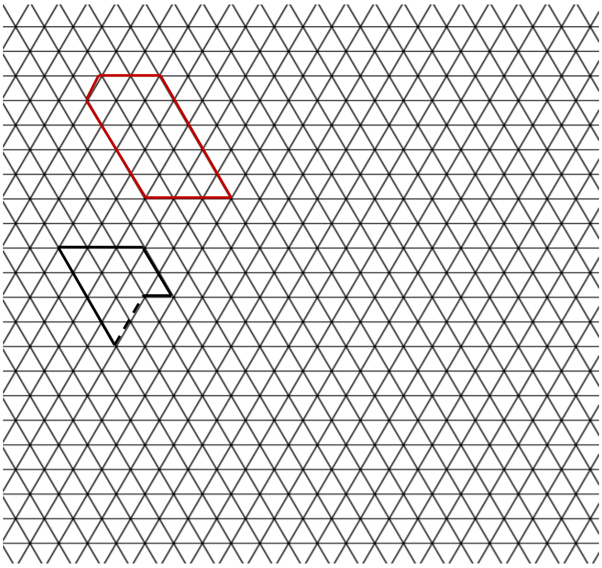

Figure 3. The triangle grid

Tiling and covering problems

In mathematics, a tiling (or tessellation) of the plane is a collection of subsets of the plane, i.e. tiles, which cover the plane without gaps or overlaps. *A shape tiles the plane* means that copies of a given shape can cover the plane, so there are no gaps or overlaps. Here we deal with few tiling problems when a tile is a magic polygon.

Task 1. Prove that there is a magic octagon (polyomino) that tiles the plane.

This is an easy task, but finding all such octagons is already a difficult task suitable for pupils as a topic for research work. The easiest way to solve this and many other tiling problems is to find a pattern that we can repeat infinitely many times. A well-known example is the golygon shown in Figure 4 (Sallows et. al., 1991). While gaining some results with golygons Sallows even explored polyiamonds and found out that perfect polyiamonds tile the plane, too. Pentagons are not unique shapes on the triangle grid tiling the plane, for example, the hexagon shown in Figure 5 tiles the plane.

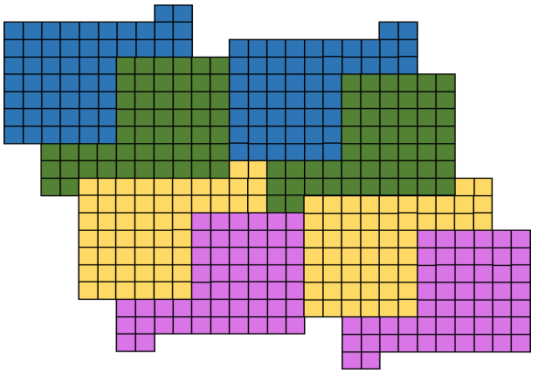

Figure 4. The golygon that tiles the plane

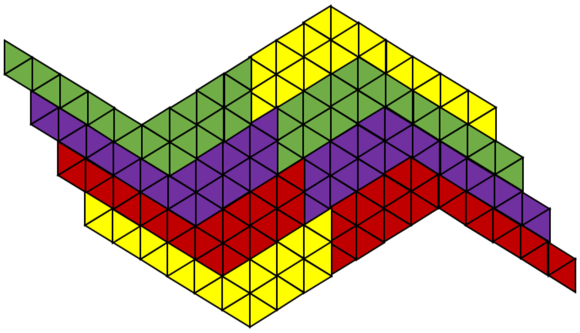

Figure 5. The magic polyiamond (6-gon) that tiles the plane

Task 2. Prove that magic octagons shown in Figure 6 tile the plane.
Note that these octagons are from each possible type mentioned in Figure 1.

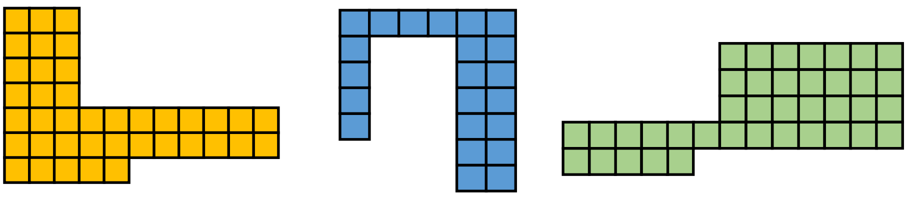

Figure 6. Some of magic octagons that tile the plane

Task 3. Do the octagons shown in Figure 7 tile the plane?

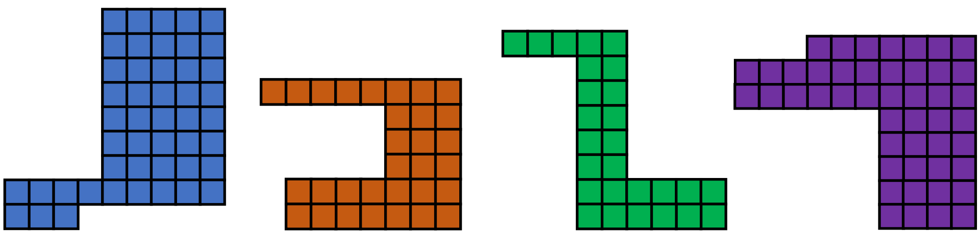

Figure 7. Magic octagons

Task 4. Prove that there is a magic octagon that does not tile the plane.
For example, a magic octagon in Figure 8 does not tile the plane since there is no way how to fill the gap it has.

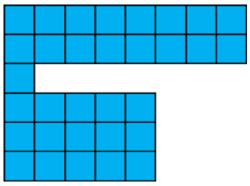

Figure 8. The magic octagon that does not tile the plane

Task 5. Prove that the magic octagon in Figure 8 is not unique that does not tile the plane. Finding all such magic octagons is a topic suitable as research work of pupils.

Task 6. Figure 9 shows how to cover the 14×14 square with 14 copies of a magic octagon. Cover the 15×15 square with copies of this octagon.
Remark. The first 14×14 square covering found by hand consisted of 20 tiles. The covering in Figure 9 was found by George Sicherman in 2019. He, by means of computer, recently found the covering of 15×15 square using 20 tiles.

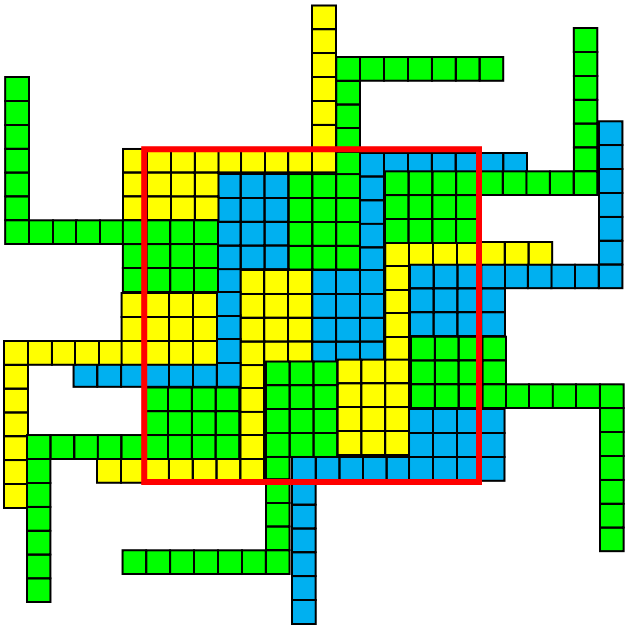

Figure 9. 14 tiles

Problem of IMO level

Let m and M be the minimum and the maximum values respectively of the areas of magic octagons on the square grid. Can the areas of magic octagons accept all integer values from the interval $[m, M]$?

To solve this problem, it is not necessary to know the exact values of m and M. Let us prove that the magic 8-gon with area 63 does not exist. How to guess this value it is already another story, cf. the remark below.
Proof. There are no magic octagons with A = 63 of the type 3 or 4. This is clear from the fact that polygons of these types fit into the rectangle 8×7. Therefore it remains to analyse the magic polygons of the first two types. Let $A(R_1)$, $A(R_2)$ be areas of the rectangles R_1, R_2 cut out of the corners of a rectangle with perimeter P = 36.

1. A magic 8-gon is of the type 1 (The first shape in Figure 1).

Depending on the length of the edge *AB*, three cases should be analysed:

1.1. If *AB* = 8 then $A(R_1) + A(R_2) = 8 \cdot 10 - 63 = 17$.

Since the lengths of the two vertical edges are: 3 + 7 = 4 + 6 then 17 can be obtained only as follows $17 = 1 \cdot 3 + 2 \cdot 7$. Cutting the rectangles 1×3 and 2×7 from the corners of the rectangle 8×10 we obtain an octagon with two equal edges.

1.2. If *AB* = 7 then $A(R_1) + A(R_2) = 7 \cdot 11 - 63 = 14$.

Since the lengths of the two vertical edges are: 3 + 8 = 5 + 6 then 14 can be obtained only as follows $17 = 1 \cdot 8 + 2 \cdot 3$. Cutting the rectangles 1×8 and 2×3 from the corners of the rectangle 7×11 we obtain an octagon with two equal edges.

1.3. If $AB = 6$ then $A(R_1) + A(R_2) = 6 \cdot 12 - 63 = 9$, but $A(R_1) + A(R_2) \geq 10$ because $1 \cdot 2 + 3 \cdot 4 = 14, 1 \cdot 3 + 2 \cdot 4 = 11, 1 \cdot 4 + 2 \cdot 3 = 10$.

2. A magic 8-gon is of the type 2. (The second shape in Figure 1).

This octagon is obtained by cutting rectangles R_1 and R_2 from opposite corners of the square 9×9, the sum of the areas of these rectangles is $81 - 63 = 18$. There are three possibilities how to get 18:

$$18 = 1 \cdot 4 + 2 \cdot 7 = 1 \cdot 6 + 3 \cdot 4 = 1 \cdot 8 + 2 \cdot 5.$$

Since 2 + 7 = 3 + 6 = 1 + 8 = 9, then the cutting out the rectangle $2 \times 7, 1 \times 6,$ or 1×8 of the corner of the square 9×9 will give a polygon having equal edges.

Remark. Problem formulation without solution is given in the paper (Cibulis, 2016). It might not be surprising that exceptional value exists, but this is a big surprise that there is only one exceptional value, and some pupils even mysteriously guess it. When pupils of Form 10 were asked to think what a value might not exist (pupils have already been solving magic octagon problems for 30 minutes), one girl mentioned that the area value 63 might not exist, because 6 + 3 = 9 is not the side length of the magic octagon. It is curious that the exceptional value 63 = 81 – 18 is the difference of two counter-readable numbers. Is there a short proof that there is an exceptional area value?

Magic polygons in practice

In the International Kangaroo Camp (Lithuania, Moletai) pupils during the lesson found 52 magic octagons and the following 36 values of area: 20, 21, 22, 23, 25, 31, 35, 37, 38, 39, 40, 41, 42, 45, 47, 48, 49, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 62, 64, 65, 66, 67, 68, 69, 70, 71. (Cibulis, 2016)

In Regional Mathematics Olympiad (School Year 2015/2016) pupils from Form 6 was offered to solve a problem: "On a square grid draw an octagon with consecutive side lengths 3, 4, 5, 6, 7, 8, 9, 10 units". Participants either solved this problem completely or did not solve at all. Only 25% of all participants did not solve the problem while almost 30% got 1 point or 2 points out off 10 points. More than 40% of pupils solved this problem completely (Avotina, Suste, 2016).

In the same Olympiad pupils from Form 10 had to solve the following problem: "Prove that for every natural *n* in squared grid it is possible to draw an octagon so that it has consecutive side lengths *n*; *n* + 1; *n* + 2; *n* + 3; *n* + 4; *n* + 5; *n* + 6; *n* + 7". Results for Form 10 are quite similar to results for Form 6 meaning that either pupils solved the problem or they did not solve it at all. About 40% of participants did not solve this problem or got only 1 point out off possible 10 points. Almost 30% solved this problem completely while there were approximately 20% that got half of the points, meaning they were not able to finish the solution (Avotina, Kokainis, 2016).

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GROUP THEORY VIA SYMMETRIES FOR ENRICHMENT CLASSES FOR GIFTED YOUTH

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Abstract. *This paper exemplifies how we design open problem fields for mathematically gifted children. We elaborate the theoretical embedding of this approach and deliver general guidelines before introducing the problem field of 'group theory of ornaments'. This is on one hand a possible working direction for students who entered the open problem field of designing ornaments, tilings or Escher's Symmetriezeichnungen, and on the other a motivation to study group theory from the perspective of symmetries. In order to do so, we motivate the abstract definition of groups via the symmetries of regular n-gons and more complex frieze patterns. We then offer an overview of possible inner mathematical follow-up topics and for detours embedding the work into culture and art.*

Key words: symmetries, group theory, ornaments, open problem fields, low floor high ceiling, mathematically gifted children, enrichment classes

INTRODUCTION

This paper follows two aims: First, introducing the mathematical area of group theory on ornaments and more generally motivating group theory from the perspective of symmetries as a great topic to be used in enrichment courses for mathematically gifted youth. Second, to reflect more generally on the question of how to find and design similar, research-oriented fields of study and corresponding work. This should yield to insights about the actual mathematical research practice.

The study of tilings or ornaments has a long tradition. We designed a version leading to aperiodic ones in Bedenknecht, Heuer and Sarikaya (MS) insofar this work can also be seen as a continuation of our previous work into a new mathematical subfield, namely group theory. Another often invoked perspective is the one from quasi-crystals, see f.i. the work of Senechal (1995). Lamotke (2005) made especially use of Frieze Groups in mathematics education and on ornaments in general consider f.i. Grünbaum & Shephard (1987).

THE DESIGN OF OPEN PROBLEM FIELDS: GENERAL THOUGHTS AND GUIDELINES.

While this paper is meant to give a more hands on introduction and an example offering a worksheet including 'solutions', we shall now give short pointers to the theoretical background: Our work is based on a tradition in Hamburg, see for instance K. Kießwetter & H. Rehlich (2005, 2008) and Nolte & Pamperien (2006, 2017a, 2017b) on the idea of open problem fields. The idea is also close to the so called 'Low floor (resp. threshold) high ceiling' tasks, see f.i. Papert (1980) or McClure, Woodham, & Borthwick (2011).

We follow their thoughts by offering a first introductory problem where the children should always be able to engage. This might be the collection of first data, i.e. examples.

This search of examples should motivate conjectures, which then can be *proved* in a fashion where we do not demand the highest standards of rigor. At the same time and especially afterwards we encourage the students to design new problem fields on their own.

We also want to stress that mathematics is part of our culture and has inherent aesthetic values. In order to do this, we implement possible detours into culture, art, history and applications to engineering or sciences. This allows students to pause mathematical activity for a while, when they get stuck and demotivated, while still engaging in the topic. This endeavour is set as a collaborative task, i.e. we allow and encourage groupwork, the exchange of results and stress the task to communicate the findings by presenting them on the blackboard in an accessible way.

To work in an open problem field might look challenging for a teacher at first glance, but we claim that it actually is a very thankful task. As we have argued in Bedenknecht, Heuer & Sarikaya (MS): The role of the teacher changes: instead of delivering answers on demand, the teacher accompanies the student in the search for answers and new problems. Since it is not clear beforehand which directions the students take, it is not possible to go through every path in advance. It is helpful to learn about differently connected fields to be able to have an association to different suggestions and attempts from the students.

In open problem fields it is normal to run into dead-ends and this increases one's frustration tolerance. Students will need help while developing heuristics of problem solving and generation. This includes the encouragement to work out examples in order to find regularities and to learn about directions to formulate new problems, like the variation of the dimension of the problem or changes of the involved parameters or rules.

WORKSHEET: GROUP THEORY VIA SYMMETRIES

Today we will have a look at symmetries, and we will learn about a mathematical tool to distinguish them. Let us first consider a piece of A4 paper on top of a desk – a rather boring situation so we might start to toss it around a little. We might ask the following question:

Problem 1: In how many ways and, especially, in which ways can we move the paper such that after the movement it occupies the same area of the desk as before the movement?

A piece of paper is a quite simple form – nothing but a rectangle. So let us ask the same question but instead of the whole piece of paper, we cut out the shape of some block letter.

Problem 2: Can you find letters which leave fewer or more options for possible movements than we had for the whole piece of A4 paper?

Let us make some remarks and collect some observations about what we have seen so far:

1. Whenever we had two movements of our paper or letters, call them m_1 and m_2, then the immediate composition of those two movements is of course also a valid movement. Let us write $m_2 \circ m_1$ for the composition of those two where we first move using m_1 and then with m_2.
2. Although not a proper movement, we could always choose to not move the paper or letter at all. Let us allow to consider this also to be a movement. We shall call it the *identity movement* and abbreviate it by writing 'id'.

3. For every movement, say m, of our piece of paper or letters there was always another movement, call it m^{-1}, which annuls m. This means that after moving with m, executing movement m^{-1}, the paper or letter is in the initial position. In other words, the composition $m^{-1} \circ m$ equals id. We call m^{-1} the *inverse movement* to m.
4. Finally, we note that the composition of our movements is *associative*. This means that for any three, maybe not distinct movements m_1, m_2, m_3 the following holds:

$$(m_3 \circ m_2) \circ m_1 = m_3 \circ (m_2 \circ m_1)$$

The name 'associative' from observation 4. is motivated from the integers. If we want m_1, m_2, m_3 to be integers instead of valid movements of our paper or letter and replace the composition operation $\circ$ by +, so the addition of integers, we get:

$$(m_3 + m_2) + m_1 = m_3 + (m_2 + m_1)$$

We call any set, say G, together with a binary operation, say $\bullet$, which satisfies the four observations above, a *group*. More precisely, we mean that the elements of G instead of the movements m_1, m_2, m_3 together with the operation $\bullet$ instead of the composition operation $\circ$ satisfy the properties above.

So, the set M of all valid movements of our piece of paper (resp. letter) together with the binary operation $\circ$ which composes two movements, is a group. We give it a special name and call it the *symmetry group of the A4 paper (or letters)*. Another example is the set $\mathbb{Z}$ of all integers together with the binary operation +, so the usual addition of two integers. We call it the *additive group of the integers*.

Now let us return to our initial problem, but again replace the rectangular shape of the A4 paper by other shapes. Let us consider pieces of paper whose shapes resemble the ones of *regular polygons*. To recall that term: we call a polygon *regular* if all edges of the polygon have the same length and additionally all interior angles have the same size.

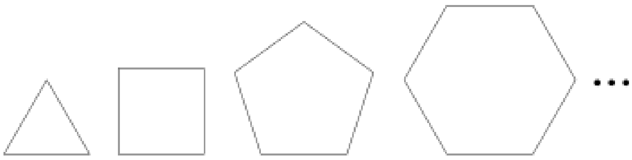

Figure 1: Regular n-gons with fixed side length.

Problem 3: Describe the symmetry group of a piece of paper whose shape resembles a regular polygon. In other words: In which ways can you move such a piece of paper so that after the movement, it occupies the same area as before you moved it?

Next let us take one step back from our comparably small pieces of paper we have considered so far and have a look at *friezes*. In architecture, friezes are a very old stylistic element to decorate facades occurring already in ancient times. More precisely, friezes are decorations with ornaments along a, mostly horizontal, strip such that the pattern of the ornaments is repetitive along the strip. An example can be seen in Figure 2.

Figure 2: An early mediaeval frieze at the Pieve di San Giovanni.[17]

Let us now try to mathematically model friezes with what we have seen so far. We can again think of the strip of the frieze along which the patterns occur as a rectangular piece of paper, but where one pair of opposite sides is very long. Let us even say that those sides are infinitely long. Also, we think of the paper strip as being transparent. So the figure is visible on both sides of the strip. In order to capture the repetitive behaviour, we think of the strip to be partitioned into rectangles, all of the same size and with the same height as the strip. In each of these partitioning rectangles we may place the same figure, resembling the ornaments. Additionally, if we know think of having a second identical strip on top of the first one and move that upper strip along the lower one by the length of a partitioning rectangle, the upper strip and its ornament figures are congruent with those of the lower one. However, if we move the upper strip by a shorter distance, the ornament figures of the upper strip will not lie congruent on top of the ones of the lower strip.

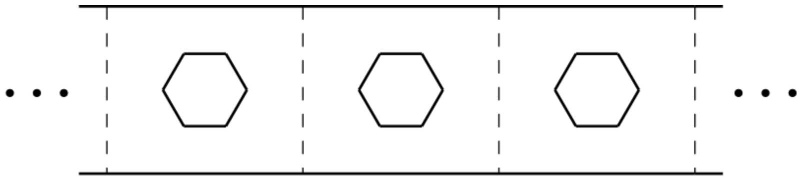

Figure 3: A schematized frieze.

Problem 4: Describe the symmetry groups that can occur for friezes. So, in other words: How can you move the frieze and its patterns such that after the movement the whole strip occupies the same area as before the movement and, additionally, all patterns together also occupy the same area as before the movement?

Think about which movements can occur and try different ornament patterns to visualise how the various groups act on the whole frieze and its pattern.

TEACHERS GUIDELINES TO THE WORKSHEET: GROUP THEORY VIA SYMMETRIES

We shall first offer solutions to the mentioned problems. We then offer other directions for follow up questions and detours into art and culture. The follow-up questions are what turns this worksheet into an open problem field. We shall only introduce possible follow-ups in a very short manner. Here the teacher comes into play, motivating the students to look for their own problems.

[17] LepoRello https://commons.wikimedia.org/wiki/File:Campiglia_Marittima_Pieve_di_San_Giovanni_northern_portal_frieze_2012-08-26.jpg , https://creativecommons.org/licenses/by-sa/4.0/legalcode

Remarks about the problems

Remark to P1: It is important to point out that each corner of the piece of paper is mapped to another one or is fixed. The same holds for the points of the plane forming the edges of the paper. Since an A4 paper does not have the shape of a square, the only movement operations are: rotation around the centre with 180°, horizontal and vertical reflection. Note that a horizontal reflection equals a rotation followed by a vertical reflection.

Remark to P2: Here the children are free to investigate, although the restriction to only look for letters does not leave much options. Here, we can find shapes allowing: **1)** no movement at all (e.g. 'L' or 'F'), **2)** only a rotation (e.g. 'S' or 'Z'), **3)** only a vertical reflection (e.g. 'T' or 'V'), **4)** only a horizontal reflection (e.g. 'B' or 'C'), **5)** the same symmetries as the A4 paper (the capital letter 'H'). However, we can find further letters with different symmetry properties, which might depend on capitalising the letter or not: **6)** while the capital letter 'X' might be written in a way to have the same symmetries as the A4 paper, the small letter 'x' is a more clear example of a letter leaving more options, namely the same as for the square, see below. Finally, we have **7)**: An exceptional letter with respect to our question is the small letter 'o' in case it is drawn as a circle. Then it leaves infinitely many movements, namely a rotation for every angle and a reflection along every straight line through the centre of the circle.

Remark to P3: The group we are looking for when considering a regular *n*-gon is precisely the dihedral group D_n. It contains 2*n* elements of which *n* belong to the subgroup generated by a rotation around the centre with the angle (*n*-2) 360°/*n*. This subgroup is isomorphic to the cyclic group on *n* elements. Then we have *n* further elements where each is a reflection along a straight line which is the bisector of an interior angle at a corner.

Remark to P4: As before, possible movements include rotation by 180° as well as horizontal and vertical reflection. Now we must additionally include translations along the strip by multiples of the length of the partitioning rectangles. Also, we might do a *glide reflection*, which means a horizontal reflection followed by a translation by half the length of the partitioning rectangles. A full classification yields 7 possible groups, called the Frieze groups. The key observation is that all Frieze groups contain all possible translations. So me might look at all ornament patterns at once, which group theoretically means to factor out the translation subgroup and consider the resulting quotient group. Now these groups are finite and contain at most 5 elements, resulting from the valid movements mentioned above. By observing that horizontal and glide reflection are never contained in the same group, we can see that the orders of the quotient groups can only be 1, 2 or 4, but with different generating elements. For the details that did not found enough place in our piece consult Lamootke (2005).

Possible follow-up questions and detours.

A classical direction to generalise questions is the already mentioned variation of the dimension of the question. A salient analogue for *n*-gons in the three-dimensional realm is the platonic solid. This allows new kinds of rotations and reflections. In general, those are harder to imagine. Here we could raise the question, which platonic solids exist, we could look for Archimedean solids. Those might also be constructed in class.

We learned about groups in this problem field. In Problem 3, we could have noticed that some groups contain subgroups. We might observe that the order (i.e. the number of elements) of the subgroup always divides the order of the group. This is actually a theorem called '*Lagrange's theorem*'. This is reachable in an advanced class with high school students. We can see that Lagrange's theorem states only a necessary condition, i.e. every order of a subgroup needs to divide the group order. This is not sufficient. Hence, we could ask which subgroups exist. A more general classification of finite groups is a huge problem field, but we could try to solve it for small group orders. Here we could look for possibilities to illustrate groups, including Cayley graphs and multiplication tables. A great detour here is to see how hard this simple sounding question actually is. The classification of the finite groups yielded to thousands of pages in mathematical journals. Other examples for groups might be studied in their own rights, like the group of the Rubik's Cube, or of more complex ornaments.

A totally different direction is it to tweak frizes generalising our problems to other ornaments and tilings. Detailed suggestions into this direction can be found in the mentioned predating work form us. This includes the design of Escher's tilings and the introduction of the notion of periodicity motivates the study of Penrose tilings. Asking about symmetry groups of those tilings leads us again to our group theoretical problems but in a higher dimension than in the case of friezes:

The most suitable kind of detours are into art and history. We can find beautiful patterns in antique architecture, Islamic mosaics, Escher's work and in Arabic rugs.

Acknowledge

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EXTENSION AND DEVELOPMENT OF DIFFERENT NON-NEWTONIAN CALCULUS IN ORDER TO SOLVE DIFFERENT DIFFERENTIAL AND DIFFERENCE EQUATIONS BASED ON MATHEMATICAL EDUCATION APPROACHES

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Abstract. *: In this paper, at first we review Newtonian calculus and some Non- Newtonian calculus's which introduced and extended in recent years. These creations and generalizations have been done by several mathematicians to reach to different goals. Then we centralize on mathematics educational approaches. After that we give some ideas and methods to obtain invariant functions with respect to related derivative and calculus. We will use these invariant functions to solve several and different differential and difference equations*

Key words: Invariant function. multiplicative derivative, Discrete multiplicative derivativ

Introduction

Additive calculus (or classical calculus) was introduced in the 17th century. This calculus is called Newtonian calculus sometimes. This calculus and its beautiful result differential equations modeled and solved many physical and engineering problems in 18,19,20 centuries (CyrillePiatecki, 2014) (Evse, 1903). The problems which are solved by this continuous calculus are appeared in continuous spaces with continuous variables.

However, there are several problems in economy and natural phenomena need to use the others and different calculus’s for solving these problems. To provide this, some discrete and continuous additive and multiplicative calculus were introduced by mathematicians. In 1972, M. Grossman and R. Katz introduced a new calculus. Afterward this calculus was called Non-Newtonian calculus (or geometric and bigeometric calculus) [3,4]. After that, D.Stanelyin (Stanley, 1999) and A.E.Bashirovet all in (Bashirov, Kurpinar, & Ozyapic, 2008)extended the multiplicative calculus in continuous case.

On the other hand, N.Aliev et all in (Aliev, Bagirov, & Izadi, 1992) studied the additive calculus in discrete case. They introduced additive discrete calculus and gave some basic formula for discrete additive derivative and integration.After that, N.Aliev and M. Jahanshahi et all in (Aliev & Jahanshahi), (Hosseini & Jahanshahi,)(Jahanshahi, Aliev, & Khatami, 2004) presented the multiplicative calculus in discrete and continuous cases.

In this paper, we introduce some invariant functions for discrete and continuous multiplicative derivatives. By using these multiplicative invariant functions, we solve linear nonlinear difference and differential equations. (Khatami, Jahanshahi, & Aliev, 2004)

We will also extend the analytical and numerical methods for solving linear and nonlinear difference and differential equations via multiplicative calculus’s.

Continuous multiplicative derivative

In this section, we introduce the continuous multiplicative derivative (CMD). We will see this new operation for derivative (is an inverse operation) is constructing by two inverse operation root and Division.

Definition: suppose the function is a positive value function, by using the definition of additive derivative as:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

We defined the CMD of function f by following limit operation:If this limit exists, then the function f is multiplicative differentiable and we show that by $f^*(x)$.

Now we try to obtain a practical formula for calculating of CMD of arbitrary function, For this according to the above mentioned definition we have:

$$\boldsymbol{f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)} - \frac{f(x)}{f(x)} + 1\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(1 + \frac{f(x+h)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+h)-f(x)}\cdot\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)}} = \lim_{h\to 0}\left[\left(1 + \frac{f(x+h)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+h)-f(x)}}\right]^{\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)}} = e^{\frac{f'(x)}{f(x)}} = e^{(\ln f)'(x)}}$$

Therefore, we obtain the following important formula:

$$\boldsymbol{f^*(x) = e^{(\ln f)'(x)}}$$

For the second order multiplicative derivative we have:

$$f^{**}(x) = e^{\left(\ln f^*\right)'}(x) = \mathrm{e}^{(\ln f)''(x)}$$

By mathematic induction we can derive for arbitrary order n :

$$f^{*(n)}(x) = e^{(\ln f)^{(n)}(x)} \;,\; n = 0,1,2,\ldots$$

From above relation, we can determine between relation of additive and multiplicative derivative:

$$f'(x) = f(x).\ln\left(f^*(x)\right)$$

Continuous multiplicative integration:

From additive integration we have:

$$S(f,p) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \qquad\qquad \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

Like continuous multiplicative derivative, we try to obtain a practical formula for calculating continuous multiplicative integration. The continuous multiplicative integration is defined by following relations:

$$P(f,p)=\int_a^b f(x)^{dx}=\lim_{n\to\infty}\prod_{i=1}^{n} f(c_i)^{\Delta x_i}=e^{\lim_{n\to\infty}\ln[f(c_1)\Delta x_1\cdots f(c_n)\Delta x_n]}$$
$$=e^{\lim_{n\to\infty}(\ln f(c_1)\Delta x_1+\cdots+\ln f(c_n)\Delta x_n)}=e^{\int_a^b \ln f dx}$$

Continuous multiplicative integration:

Suppose thefunction f is defined by:

$$f:A\subset Z\to R,\ x\in A\subset Z$$

then the discrete multiplicative derivative (DMD)of function $f(x)$ is defined by:

$$f^{[1]}(x)=\frac{f(x+1)}{f(x)}$$

we show that by $f^{[1]}(x)$.

Example1.For the power function $f(x)=x^n$ we have:

$$f^{[1]}(x)=\frac{f(x+1)}{f(x)}=\frac{(x+1)^n}{x^n}=(1+\frac{1}{x})^n$$

Example2.For the exponential function $f(x)=a^x$ we have:

$$f^{[1]}(x)=\frac{f(x+1)}{f(x)}=\frac{a^{x+1}}{a^x}=a$$

Therefore, the DMD of exponential function is its basis.

Invariant function :

It is easy to see that the invarient function for this kind derivative is $f(x)=c^{2^x}$. In fact, we see that

$$\left(c^{2^X}\right)^{[1]}=\frac{c^{2^{X+1}}}{c^{2^X}}=c^{2^{X+1}-2^X}=c^{2^X(2-1)}=c^{2^X}$$

5. Discrete Multiplicative Integration

In this section , we give the concept of multiplicative discrete integration of a function . For this, suppose for the two functions f and g , we would have the following relation, [26],[12]

$$f^{[1]}(x)=g(x),$$

where the notation [1] is used for multiplicative discrete derivative.

According to the definition of Multiplicative derivative , we can write:

$$\frac{f(x+1)}{f(x)}=g(x),$$
$$f(x+1)=g(x)f(x),\qquad x\ge x_0.$$

If we begin from the point x_0 and with $h=1$, we will have:

$$f(x_0+1)=g(x_0)f(x_0),$$
$$f(x_0+2)=g(x_0+1)f(x_0+1),$$
$$\vdots$$
$$f(x_0+n)=g(x_0+n-1)f(x_0+n-1).$$

Now by multiplying these relations together side-by-side, we obtain:

$$f(x_0+1)f(x_0+2)\cdots f(x_0+n) = g(x_0)g(x_0+1)\cdots g(x_0+n-1) \cdot f(x_0)f(x_0+1)\cdots f(x_0+n-1).$$

If we remove the expression

$$f(x_0+1)f(x_0+2)\cdots f(x_0+n-1),$$

from each sides of above relation , we have

$$f(x_0+n) = f(x_0)\prod_{j=0}^{n-1} g(x_0+j).$$

If we use the notation $\int$ for this kind of integration , we can write

$$f(x) = c\int_{x_0}^{x} g(\xi) \equiv c\prod_{\xi=x_0}^{x-1} g(\xi).$$

Discrete Multiplicative Differential Equation

Remember the function $y(x)=c^{2^X}$ as an invariant function, for discrete multiplicative differentiation. We saw that:

$$y^{[1]}(x)=\frac{c^{2^{X+1}}}{c^{2^X}}=c^{2^{X+1}}c^{-2^X}=c^{2^X(2-1)}=c^{2^X}$$

it means the derivative of function is equal to itself. We use this function to construct the general form of the solutions of DMDE’s, that is:

$$y=C^{(\lambda+1)^X}=C^{\left[(\lambda+1)^X\right]} \qquad (1)$$

where C is a constant and λ is a parameter.

We calculate the first, second and n-th order derivative of this function:

$$y^{[1]}(x)=\frac{C^{(\lambda+1)^{X+1}}}{C^{(\lambda+1)^X}}=C^{((\lambda+1)^{X+1})-((\lambda+1)^X)}=C^{(\lambda+1)^X}.C^{(\lambda+1)^1}.C^{-(\lambda+1)^X}$$

$$=C^{(\lambda+1)^X(\lambda+1-1)}=C^{(\lambda+1)^X.\lambda}$$

Similarly, for the second derivative, we have:

$$y^{[11]}(x)=C^{(\lambda+1)^X.\lambda^2}$$

and by mathematical induction we get the following general formula

$$y^{[n]}(x)=C^{(\lambda+1)^X.\lambda^n} \qquad (2)$$

We begin from the first order homogenous DMDE with initial condition :

$$y^{[1]}(x)=y^a(x)\ , \qquad x>x_0\in Z \quad , \quad a=constant \qquad (3)$$

$$y(x_0)=y_0$$

by using the introduced invariant function (1) the general solution of equation (3) is:

$$y=C^{(a+1)^X}$$

by imposing the given initial condition we have:

$$C^{(a+1)^X} = y_0 \qquad \Rightarrow \qquad C = y_0^{\frac{1}{(a+1)^X}}$$

Now we consider the general form of second order homogenous DMDE:

$$y^{[11]}(x) = \left(y^{[1]}(x)\right)^a \left(y(x)\right)^b \quad , \quad x > x_0 \qquad x, x_0 \in Z \ , a, b = constant \quad (4)$$

with initial conditions:

$$y(x_0) = y_0 \quad , \quad y^{[1]}(x_0) = y_1$$

with some algebraic calculations we obtain the solution of initial value problem (4):

$$y(x) = (y_0^{\lambda_2} y_1^{-1})^{\frac{(\lambda+1)^{X-X_0}}{\lambda_2-\lambda_1}} \cdot (y_0^{-\lambda_1} y_1)^{\frac{(\lambda_2+1)^{X-X_0}}{\lambda_2-\lambda_1}} \quad , \quad \mathrm{x} \in Z$$

Continuous Multiplicative Differential Equations

The invariant function with respect to continuous multiplicative derivative is the function

$y = e^{e^X}$.

We will use this invariant function to solve continuous multiplicative differential equations (CMDEs). The general form of first order CMDE can be shown by:

$$x^*(t) = f\,(t, x(t)) \qquad (3)$$

where x(t) is unknown function and ∗ denotes the multiplicative derivative .The second order CMDE

$$x^{**}(t) = f\,(t, x(t), x^*(t)) \qquad (4)$$

can be given with two initial conditions:

$$x(t_0) = x_0 \quad , \quad x^*(t_0) = x_0^*$$

The existence and uniqueness of solutions of CMDEs is same as ordinary differential Equations, [6,7][1819]. There are limited methods to solve analytical. Similar to ordinary differential equations, for the first order CMDE, we try to use multiplicative integration

Example1. For solving the first order CMDE $y^* = xy$ we can write its solution by multiplicative integration:

$$y(\mathrm{x}) = \int ty^{dt} = e^{\int^x \ln ty(t)dt}$$

This is an integral equation which can be solved by multiplicative methods.

Example2. Consider the following non-linear ODE:

$yy'' - y'^2 = y^2 x \quad ; (5)$

At first, this equation is rewritten as follows:

$$\frac{y''y - y'^2}{y^2} = x \Rightarrow e^{\frac{y''y - y'^2}{y^2}} = e^x \Rightarrow y^{**}(x) = e^x$$

To solve this multiplicative differential equation, it is enough to apply twotime multiplicative integration, that is:

$$y^*(x) = c_1 \int^x (e^t)^{dt} = c_1\, e^{\int^x \ln e^t dt} = c_1 e^{\frac{x^2}{2}}$$

$$y(x) = c_2 e^{\int^x \ln c_1 e^{\frac{t^2}{2}} dt} = c_2 e^{\int^x \left(\ln c_1 + \ln e^{\frac{t^2}{2}}\right) dt} = c_2 e^{x \ln c} e^{\frac{x^3}{3!}}$$

hence the two independent solutions are :

$$y_1(x) = c_1 e^{\frac{x^3}{3!}}$$

$$y_2(x) = e^{x \ln c + \frac{x^3}{3!}} = e^{x \ln c} . e^{\frac{x^3}{3!}} \qquad (6)$$

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PÓSA METHOD: TALENT NURTURING IN WEEKEND MATH CAMPS

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Abstract*: Lajos Pósa has been organizing weekend mathematics camps for highly gifted students to foster their development with a special method since 1988. During these 30 years, he and his disciples led more than 350 camps for more than 1500 students. Students' work dominantly takes the form of a special team work, and is based on a five-year-long coherent curriculum organized around problem threads, which form a complex web. These threads run parallel, in 'harmony', supplementing and assisting each other's development. Effectiveness of the camps is reflected in the fact that in the past 25 years, almost all members of the Hungarian IMO teams were participated in these camps before.*

Key words: mathematically gifted learners, learner autonomy, discovery learning, inquiry based mathematics education, connected task-design, problem thread, teamwork

BACKGROUND AND THE BEGINNINGS OF THE PÓSA CAMPS

In the beginning of the 1970s, Lajos Pósa joined the "complex mathematics education" movement in Hungary led by Tamás Varga (Halmos & Varga, 1978), having also been inspired by the ideas of many influential Hungarian mathematician and mathematics educators of the middle of the 20th century, like the work of György Pólya on problem solving (Polya, 1957). In the 1980s, Pósa became a member of the group at the Rényi Institute of Mathematics, led by János Surányi, studying the applicability of the Varga method in high schools. Between 1982 and 1991, he taught 2 high school classes for 4 years each, during which he developed teaching materials with the aim of making the learning of mathematics enjoyable, and focusing not on mechanically applying formulas and algorithms, but on relaxed and cheerful, autonomous and logical thinking.

He then turned to talent care, which has had a well-established tradition in Hungary. There were a national network of special mathematics circles for students and several 'high-quality' competitions were (and are) organized, as well as a mathematics journal for students, the KöMaL has been published since 1893, currently also available in it's website (http://www.komal.hu/info/bemutatkozas.e.shtml). However, Pósa felt that the school environment, the short learning sessions (maximum 90 minutes study circle sessions) do not enable students to be involved in the intense thinking that highly talented students need. Besides he did not consider preparing for competitions to be at the core of nurturing talent, but rather that students shall think on exciting, interesting problems and discover the beauties of mathematics autonomously.

Pósa launched a talent nurturing weekend math camp for one single group in 1988. The first author was a member of this group of roughly 15 students. Camps' schedule had been evolved continuously, in these early times the main activity was students' individual autonomous work, together in a large room. While in these first camps Lajos Pósa was the only teacher and organizer, there are presently 2-7 'assistant teachers offering help to the 'camp leader'. For more details, see (Győri & Juhász, 2018)

PARTICIPATING STUDENTS

Hungary has an elaborate system of mathematics competitions, with individual and team ones, multiple choice tests and ones requiring detailed written solutions or proofs. Based on the results of the most prestigious competitions, and recommendations of teachers and parents we invite approximately 60 students (grade 7) to the beginner groups of our math camp. As from 2010 we launch 2 beginner groups at grade 7 each year, which allows us to invite approximately 60 students altogether, there are some other, much less exact invitation methods, based on teachers' and parents' recommendations. In these cases, students have to solve a set of 'entrance' problems, as we think that participating in the camps among peers being substantially 'better' is not good for the students.

All camps are free for all students, so that no gifted student left out for financial reasons. There is a continuously growing demand for participation.

There are 10-12 groups running parallel at the same time (1 or 2 at each grades 7-12), that is roughly 300-350 students are involved. We organize yearly 2.5 camps for one group, in average, and the groups usually have 10-13 camps from grade 7 until grade 11 or 12.

A TYPICAL CAMP – WHAT DOES IT LOOK LIKE?

A typical camp starts at 5 pm on Friday, and ends at 3 pm on Sunday, with a total of 13-15 full hours spent purely on math, at least half of which is devoted to students' autonomous thinking. The mathematical work can typically be categorized into 4+1 types of situations: 1) 'Autonomous Thinking' in small groups, with special rules (see section on Special Team Work); 2) 'Plenary Discussion' on problems, solutions and conclusions; 3) 'Individual Thinking' (typically) on 'lightning round' questions (with short time limit), physically the whole group together; 4) 'Team Contest': autonomous thinking in small teams with different rules and partially different aims than that of 'Autonomous Thinking'; +1) 'Homework', with problems to be thinking on between two camps.

Type 1) will be discussed in details in the section on Special Team Work.

During 'plenary' discussions (type 2) we talk about homework and those problems that have been solved by enough students during autonomous group thinking. Students present their solutions, often several different ones (if there are). It is prominently crucial that students also pose new questions in view of the discussed solutions. As a significant part of our pedagogy, we consider students' learning to pose good questions very important. Posing questions are rewarded by chocolate and students regularly vote for student-posed new questions to be built into the camp's curriculum, as they wish to think them on. These questions are then named after the students who posed them. Over the years, camps' curriculum is being broadened by many student-posed problems. Discussing problems is, at times, supplemented by 'tales' about connected moments in the history of mathematics or current problems and results in mathematical research.

Once in every camp, in average, students think on problems individually, but at the same room (type 4). These are usually blocks for 'lightning round' questions', with the aim of evoking previously discussed topics or consider it from a different point of view. These may also be loaded questions (trick questions), trying to lure solvers into traps, in order for calling attention to the limits of certain previously used methods or techniques, that they can only be used under certain circumstances. High numbers of good answers to 'lightning round' questions are also rewarded by chocolate.

Saturday afternoons are always devoted to the team contests (type 3), where the special rules of special team work (type 1) do not apply, instead the teams work with the widespread methods of 'group work'. Teams need to do their best in solving 5 problems in usually 2.5-3 hours. These problems are special in that teams also compete with each other problem by problem (not only on the whole). They need to find the sole different weight with as few number of weighting as they can (balance puzzles); select as many elements from a set, with a particular property to be hold, as possible; construct a triangle with a maximized area or perimeter, under certain boundary conditions; solve a certain problem with the lowest cost possible; etc. Initially, there were no team contests in the camps, as competition is (almost) the last thing we connect to the real nature of doing mathematics. Instead, we regard it as mainly characterized by relaxed and deep thinking, in which time does not play an important role. Moreover, we prioritize the importance of each individual's thinking on problems appropriate to their own 'level', and that they shall, first of all, compete with themselves, and not with others. However, it turned out that children are energized by these team contests in an extremely intense and positive way, so it has become an integral part of the camps, but still not significant in time, also rewarded by chocolate.

SPECIAL TEAM WORK

In the Pósa method, the main focus is on students' autonomous thinking, that as many students shall discover as many ideas autonomously as possible. This may support the need for a dominance of students' individual work. However, our experience has shown something different. In the first camps, students were mainly thinking individually, together at the same place; keeping track of each other's progress, as their competitive spirit made them being interested in which problem was solved by whom. Faster performing students tended to work less hard, after realizing their relatively outstanding achievements; while - which we consider a more serious problem - slower performing students were discouraged, they had lost their interest in working on, after realizing their relatively slow progress. We, in turn, think time (speed of progress) does not play such an important role. That is the reason why Pósa has changed the main form of students' activity into thinking in groups of 2-4, in separate rooms, having no information on the other groups' progresses.

Being the whole camp together, usually 5-8 problems are posed at the same time, and then the teams go to their separate places 'alone'. The recommended structure of team work is the following: 1) each student think individually on each problem (with a freedom of choosing the order); and if they think they have solved one, they shall not tell it to their peers who don't; 2) if a problem is considered to be solved by all team members, they can discuss their ideas; 3) if a number of them has been struggling with solving the same problem, without much results, than they can brainstorm together. However, during stage 3), if a student discovers a crucial idea leading to the solution, they need to quit the discussion, allowing their peers to discover it too. In grade 7, this method does not work smoothly, as students usually do not understand why they are not free to talk when working in groups. It depends on the individual students, and the particular groups when they start to realize that this 'rule' is for their own interest; however, until the second term of grade 8, this system is usually well developed. Around grade 10, many students go to the other extreme, and though teams exist, they insist on solely autonomous work with a decreased wish for cooperation, they want to solve each problem themselves. However there are some, who continue doing the 'restricted cooperation' (or 'limited group work').

FEATURES OF THE PROBLEM SET – THE WEB OF PROBLEM THREADS

Research on reconstructing the method in a theoretical level

The Pósa method as a complex teaching practice with special task-design and classroom management elements is based in "teaching as craft knowledge" (Watson & Ohtani, 2015, p. 5) and has never been completely developed on a theoretical level with a conceptual basis of Pósa's didactical conceptions. From 2016, research is being carried out by the second author mainly to reconstruct the theoretical model behind the method as an "intermediate-level framework (Watson & Ohtani, 2015, pp. 19-81) or as a variant of already established frameworks, such as the Anthropological Theory of the Didactics (ATD) (Bosch & Gascón, 2014), so that the concepts of our introduced WPT model (see next paragraph) is studied and explained by ATD tools, such as praxeological analysis (Katona, 2018, January).

Our research is also to contribute to the international discourse on conceptualizing Inquiry Based Mathematics Education (Artigue & Blomhøj, 2013; Bosch & Winsløw, 2016). As a first step of theorizing, a mathematical content analysis was conducted, focusing on the connection between the problems, as well on the attribute of the learning goals of the Pósa method that they are essentially realized in long-term. This resulted in introducing the term and 'web of problem threads' for a particular task-design and curriculum development tool (Katona & Szűcs, 2017).

PROBLEM THREADS – A CONNECTED TASK-DESIGN TOOL FOR (A TYPE OF) INQUIRY-BASED LEARNING MATHEMATICS

Problems posed at the Pósa camps are connected to each other in multiple ways. A set of connected problems, with a not completely fixed order, is called a 'problem thread'. These connections manifest themselves in the form of common features of the problems, that they foster the development of specific ways of mathematical thinking, or methods, called the 'kernels of the threads'. Although these kernels are usually not of traditional, content-based categorizations, that is, threads are typically not created by problems just belonging to the same content areas, such as geometry or number theory, there are some content-based threads, like 'remainders', as we think that modular arithmetic and it's effective application in problem solving shall be clear to every student. Threads typically cover many mathematical content areas. Some important kernels are: 'recursive thinking', '(proof of) impossibility' and 'movement'. These threads run for years; there are threads which are part of the curriculum from the first camp until the last one. Students' attention is also drawn frequently to the connections between the problems, we regularly ask in which task they have met a similar line of thought before. Students are asked to tag the problems by the names of the kernels for categorizing them.

Threads are not separated. They regularly cross each other at problems belonging to these crossing threads, forming the 'web of problem threads' (WPT) (Katona & Szűcs, 2017). Astonishing moments of the camps are created by solving the problems that connects two or more threads. Students then realize that in mathematics, apparently remote areas may also be closely connected, and finding solutions may be facilitated by ideas that we may not think of right in the beginning. Threads run simultaneously, crossing each other at the appropriate moments, then go on. Sometimes we do not meet a thread for a longer time, until it comes back.

Using Pósa's allegory it's like a well-composed piece of polyphonic music. Phrases can be heard simultaneously, perfectly supplementing each other. At time, some phrases are quieter, later may be louder, contributing to an ideal musical experience. However, the composition has and will never be finished. The WPT is being developed from the beginnings, and has an inherently dynamic nature. Problems may disappear, for some time (for some groups of students); new ones appear regularly, often posed by students.

EXAMPLE FOR A PROBLEM THREAD

There are a lot of different problem threads in Pósa method. Here we introduce a tiny one to understand what a thread means.

Problem 1. Does there exist an arithmetic progression with six terms so that each pair of terms is relatively prime? (The terms of the AP are distinct positive integers.)

After the students solved this problem we ask them to pose good questions. The following questions are the most typical ones:

Problem 2. What is the maximum length of an AP if each pair of terms is relatively prime?

Problem 3. Does there exist an infinite AP so that each pair of terms is relatively prime?

Some students recommended the following questions:

Follow-up Q1. Does there exist an infinite sequence so that the differences of the consecutive terms form an AP and each pair of terms is relatively prime?

Follow-up Q2. If the terms of the AP are primes then it implies that each pair of terms is relatively prime. Therefore a naturally arising question is the following: Does there exist an arbitrarily long AP whose terms are primes? The answer for this question is very hard. It is yes, and it was proved by Ben Green and Terrence Tao in 2004 (Green & Tao, 2008).

GOALS, EFFICIENCY AND RESULTS

No controlled measurements on the efficiency of the Pósa method have been conducted so far, as we aim at assisting the development of such abilities that are extremely difficult to be measured; efficient and trustworthy method for measurement are not known (by us).

One targeted 'ability' to be developed is that students shall love thinking hard and persistently on difficult and extremely difficult problems. Another one is, that in case of initial failures, they shall not give up, and shall approach the problem from several different 'directions'. The third one is that although the curriculum intends to teach 'thinking methods' (spinned into the WPT), these cannot always mechanically and directly be applied to new problems. The set of these methods is a tool bar to be applied in an intelligent and creative way, sometimes with improved variants of the methods. The fourth and perhaps most important one is that students shall be happy while doing 'high-quality' mathematics.

Based on students' oral feedback, our camps are very successful. Many of them report that the way of thinking they have learnt in the camps help them a lot in their work, which is not infrequently far from doing pure mathematics.

Fostering the development of students' competition problem solving skills is definitely not targeted in our camps. Still, students' feedback indicates that, during the math competitions, they can boneneficially apply their knowledge learnt in the camps. In the past 25 years, with only some exceptions, almost always all the 6 members of the Hungarian IMO teams came from the Pósa camps. There is a similar situation for the 6 member MEMO teams (Middle European Mathematical Olympiad), from its starts in 2007. The following table contains data about the participating Pósa students in the last 10 years.

	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
IMO	6	5	5	6	6	6	6	6	6	5
MEMO	5	5	5	6	5	4	6	4	6	6

Table 1: Number of 'Pósa students' in the Hungarian IMO and EMO groups, 2009 - 2018

It is important to note that the Pósa camps are not part of the official IMO and MEMO participatory program in Hungary. Our students have outstandingly good results in the national competitions too. Many of them continue their studies at the world's leading universities.

FURTHER RESEARCH

We believe the principles of the Pósa Method can be used in various settings. One of our further, ongoing research foci is on public education, we started a four-year experiment with three groups of grade-9-students, we are interested in what way the Pósa method can be used in a normal high school setting. Our other research interest is about how to reach out to talents. Our "Flying School" program offers mostly underprivileged high schools a 3-4 hours inspiring activity for grade 9 students. For the talented students we offer a further opportunity of a two-year long talent-nurturing program, organizing them a "mathematical" day every months.

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MATHEMATICS EDUCATION PRE-SERVICE TEACHERS AWARENESS OF GIFTED STUDENTS' CHARACTERISTICS

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Abstract: *Gifted students present an array of characteristics in mathematics. However, in South Africa most of the tertiary institutions do not cater for gifted education to provide opportunities for pre-service teachers to be abreast of the mathematically gifted students' characteristics. This study investigates pre-service teachers' a**wareness** of mathematically gifted students' characteristics. This is done to strengthen the claim to introduce a module on gifted education in Mathematics Education programme at the Central University of Technology (CUT), in South Africa. The study followed a qualitative and quantitative approach where sixty-six pre-service teachers' responses were collected and analysed. The results revealed that pre-service teachers' have limited acquaintance regarding the characteristics of gifted students in mathematics. Thus, the inclusion of modules on gifted education in the mathematics training program is needed.*

Key Words: *Gifted education; Mathematically gifted; pre-service teachers, gifted students' characteristics.*

INTRODUCTION

In order to understand the need for giving a special attention to teaching the gifted student it is important to know that in South Africa most of the gifted students study mathematics in regular classrooms. Benson (2002) observes that gifted learners become more frustrated in a regular classroom while teachers are mandated to see that all learners reach the standards of the Provincial and National Department of Education. However, in South Africa teachers do not receive any special training on how to identify gifted students' characteristics to address their need in regular classrooms. Awareness of these characteristics will help teachers with the early identification process (DoE, 2010: p. 9). Therefore, it is important for all pre-service teachers to learn about gifted students' characteristics during their training. Investigating pre-service teachers regarding their awareness of gifted students' characteristics is expected to provide valuable information on the aspects which are susceptible of improvements. In addition, this study could serve as a starting point for the development of training program for pre-service teacher concerning mathematically gifted students.

Problem statement

Although abundant research documents for specific needs of gifted students, too little has been done in South Africa to prepare pre-service teachers to be aware of gifted students' characteristics and meet their needs in mathematics. There is a need to focus attention on and gain understanding of giftedness. Authorities in the field of gifted education maintain that all educators working with gifted students should receive adequate training in the characteristics and needs of this special population in order to meet their specific needs (Feldhusen, 1997; Gallagher, 2000; & Toll, 2000). Research with this focus therefore seems vital. This study contributes in the inclusion of modules on teaching gifted students in mathematics education training program at CUT.

CONCEPTUALIZATION OF GIFTEDNESS

There are various definitions of a gifted students. Some put emphasis on the student's current level of achievement based on an overlap and interaction among three clusters of traits: above average ability, task commitment, and creativity (Renzulli 1986); whereas for others, the key is the child's potential to perform at a level significantly beyond age-peers (e.g. Gagné, 2003). Gagné (2015) was concerned about treating gifted learners as belonging to a homogenous group arguing that there are different levels of giftedness. According to Gagné's (2005) giftedness is associated with the outstanding natural or innate abilities. Gifted individuals should have natural abilities at least to the degree of the top 10 percent of age peers. In the same way, Renzulli (2012) used the terms 'high achieving' or 'schoolhouse giftedness' to refer to students who are good lesson learners in the traditional school environment. Thus, in this paper the term 'gifted learner' is used to refer to the 'mildly gifted' in accordance with the recommendations from Gagné (2015).

Characteristics of mathematical Giftedness

Numerous models of giftedness consider general abilities and domain-specific abilities as important for optimal achievement in a specific field (Subotnik, Olszewski-Kubilius, & Worell, 2012). General abilities in academic domains are often defined by abilities measured by intelligence tests. As noted by the term, domain-specific abilities need to be described in relation to the specific field of giftedness. This study focuses on the field of mathematical giftedness based on Krutetskii's twelve-year study. According to Krutetskii (1976), "mathematical giftedness" is the name given to a unique aggregate of mathematical abilities that opens up the possibility of successful performance in mathematical activity. Krutetskii defined ability as a personal trait that enables one to perform a given task rapidly and well, and contrasts this to a habit or skill, which relates to the qualities or features of the activity a person is carrying out.

Following the basic stages of obtaining, processing and retaining mathematical information, Krutetskii (1976) identified abilities including the ability to a) grasp a formal structure of a problem, b) think logically in spatial, numeric and symbolic relationships, c) rapidly and comprehensively generalize mathematical material, d) curtail mathematical reasoning processes, e) be flexible with mental processes, f) strive for clarity, simplicity and rationality of solutions, g) to reverse and reconstruct mental processes, and h) memorize schemes, methods, principles and relationships. According to Krutetskii (1976) these abilities are interrelated and form a general synthetic component, "a distinctive syndrome of mathematical giftedness, the mathematical cast of minds". This might be interpreted as a tendency to view the world through a mathematical eye. It is also to be noted that Krutetskii does not equate mathematical giftedness with high achievements in school mathematics. This is also true for other later researchers (see e.g., Diezmann & Watters, 2002; & Pettersson, 2011). Bicknell and Holton (2009) noted that mathematical giftedness can be manifested in three ways, namely the analytic type, the geometric type and the harmonic type. The analytic type is characterized by abstract patterns of thoughts and well-developed verbal-logical skills combined with less well-developed abilities to visualize the subject. The geometric type is characterised with well-developed abilities to visualize the subject, that complement less well-developed verbal-logical skills. In this type gifted students will prefer to use sketches and visual aids to figure problems. The harmonic type, which presents the gifted students who are capable to use both ways mentioned above, the analytic and the geometric (Krutetskii, 1976, pp. 317–329).

Research Questions

Considering the literature review presented in the section above, two main questions that this research inquires to answer arise:

1. What is pre-service teachers' knowledge about:
   1.1 Their training to teach gifted students?
   1.2 Attending school with gifted students?
2. How aware are the pre-service teachers towards the characteristics of gifted students during instructions?

METHODS

Research design

This study adopted a qualitative and quantitative approach, designed, whereby pre-service teachers' responses were analysed.

Sample and Sampling Technique

The population consists of Mathematics Education pre-service teachers from the Central University of Technology, in South Africa. A purposeful sampling was used to select sixty-six (n = 66) pre-service teachers.

Research Instrument

A questionnaire with closed ended and open-ended questions was used as a tool to collect data in this study. The questionnaire was divided in three sections, namely: Bibliographical information, teachers' training and the awareness of gifted students' characteristics.

Results and Data analysis

Tables 1, 2 and 3 show pre-service teachers' responses about "teacher training" (question 1.1), "Attended school with gifted students" (question 1.2) and about "the awareness of gifted students" (question 2).

QUANTITATIVE ANALYSIS OF PRE-SERVICE TEACHERS' RESPONSES TO CLOSE-ENDED QUESTIONS

Teacher training

In ***question1. 1*,** pre-service teachers responded with "agree, neutral and disagree" about training to teach gifted students. Table 1 indicates pre-service teachers receiving training on how to teach mathematically gifted students.

Answer	**Percentage (%)**	**Count (N)**
Agree	32%	21
Neutral	42%	28
Disagree	26%	17
Total	100%	66

Table 1. Pre-service teachers' training to teach gifted students

It is evident from Table 1 that the responses to ***question 1.1***, confirm that 32% of pre-service teachers agreed that they are training to teach gifted students; 42% of pre-service teachers are neutral about training to teach gifted students; and 26% of disagree that they are training to teach gifted students. This analysis indicates that almost two third of pre-service teachers are not training to teach mathematically gifted students.

Attended school with gifted

In ***question 1.2***, pre-service teachers responded with "agree, neutral and disagree" about attended school with gifted students in mathematics class. Table 2 displays pre-service teachers' responses to this question.

Answer	**Percentage (%)**	**Count (N)**
Agree	73%	48
Neutral	20%	13
Disagree	8%	5
Total	100%	66

Table 2: Attended school with gifted students

It is obvious from Table 2 that the responses to question 1.2 have 73% of pre-service teachers responding with "agree" and 20% of pre-service teachers with "neutral" and 8% of pre-service teachers with "disagree". It is a matter of concern that a third of pre-service teachers were not able to identify gifted students or have not attended school with gifted students.

QUALITATIVE ANALYSIS OF PRE-SERVICE TEACHERS' RESPONSES TO OPEN-ENDED QUESTIONS

The pre-service teachers are referred to as *PST 01*, *PST 02*, etc. From the thematic analysis of ***question 2***, the research question was divided into ten categories: *active participation in the classroom, complete activities very fast, ask different questions, prefer working alone, prepare for class, get bore, challenge the teacher with mathematics, skilful in mathematics, disturbing/misbehaving and achievement.*

PST 01, and *PST 66* mentioned the first sub-category, *students are active in the classroom*. *PST 01* stated: "Students who are gifted are always hyperactive during class". *PST 66* stated: "A gifted student is mostly active in class and are able to do problems before the teacher starts the topic".

In terms of the second, *students' complete activities very fast*, *PST 07* stated: "A gifted student understand and grasp what the teacher says fast". *PST 18* mentioned: "They are quick in responding the correct answers".

In this third sub-category, *students ask different questions*. *PST 05* revealed: "Gifted student asks different questions in class". *PST 52* mentioned: "Gifted students always ask questions for better understanding".

In this forth sub-category, *students prefer working alone*. *PST 18* mentioned: "They can do things on their own without the teacher's help". *PST 43* believed: "is able to do well without the help of teacher".

PST 03 and *PST 04* mentioned the fifth sub-category, *students are prepared for class. PST 03* related how she/he identify gifted students: "they are always prepared for class and know everything even before the teacher can do them in class". *PST 04* mentioned: "They can work out the problem way before the teacher has done explaining to everyone in class".

In terms of the sixth sub-category, *students get bore in class. PST 03* indicated: "They get bored easily if the teacher is incompetent". *PST 13* talk about: "Gifted students give answers beyond my expectations, becomes easily bored and disrupts lesson".

In the seventh sub-category, *students challenge the teacher with mathematics: PST 05* bring out: "They like to challenge teacher with math". *PST 07* mentioned: "A gifted student identifying the teachers' mistakes".

In terms of the eighth ninth sub-category, *students are skilful in mathematics. PST 47* revealed: "they are those learners who find difficult maths problems easy for them. *PST 50* declare: "They can grasp abstract maths concepts faster than other learners".

In the ninth sub-category, *students are disturbing/ misbehaving in mathematics class: PST 34* indicated: "a gifted student always finishes their work first and start misbehaving or interrupting the class". *PST 39* point out: "They finish their work very quickly within a given time and start disturbing others".

In this last sub-category, *PST 01 and PST 09 mentioned students' achievement. PST 01* mentioned*: "*Students who are gifted always got outstanding marks in their tests or exams". *PST 09* said: "Always score high marks".

This section shows that there are pre-service teachers who have a slight idea on mathematically gifted students' characteristics such as: active in the classroom, complete activities very fast, ask different questions, prefer working alone, prepare for class, get bore, challenge the teacher with mathematics, skilful in mathematics, disturbing/misbehaving and achievement.

DISCUSSION

In the quantitative analysis, most of pre-service teachers have not received training to teach gifted students. It is a matter of concern, that a third of the pre-service student mentioned that they have been trained to teach mathematically gifted students. This controvert the CUT mathematics program where there is no teacher training for gifted students (CUT, 2019).

Results from the quantitative analysis also highlight that most of the pre-service teachers attended school with gifted students. It is alarming that a quarter of pre-service teachers disagreed or were neutral. This shows that pre-service teachers are not conversant with definitions of giftedness (Gagné, 2003 & Renzulli, 1986) and are not fully aware towards the characteristics of mathematically gifted students (Krutetskii, 1976).

In the qualitative analysis of the open-ended question, the results showed that pre-service teachers have a limited knowledge about characteristics of mathematical Giftedness (Krustetskii, 1976). Pre-service teachers equate mathematical giftedness to achievement in mathematics which contradict Krustetskii (1976); Diezmann and Watters (2002); and Pettersson (2011).

In our discussion the pre-service teachers need to understand the mathematical abilities that open the possibility of successful performance in mathematics activities (Gagné, 2003; Renzulli, 1986; & Krutetskii, 1976). According Plunkett and Kronborg (2011) the awareness of characteristics of mathematically gifted students has an impact on their identification. In South Africa, teachers concurred that they had never received training on how to identify and support gifted students (Oswald & de Villiers, 2013).

CONCLUSION

Numerous models have been conceptualised on how to improve the education of the mathematically gifted learners, in South Africa one might ask how pre-service teachers can be aware of the mathematically gifted students' characteristics. This study investigates pre-service teachers' a**wareness** of mathematically gifted students' characteristics. Therefore, the inclusion of modules on gifted education in the mathematics training program is needed.

Acknowledgments

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SIMPLE BUT USEFUL TASKS WITH GEOMETRICAL CONTENT AND CREATIVE FLAVOUR

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Abstract. *Nowadays nobody would argue the usefulness of geometrical content. Similarly nobody will say something against the thesis that geometry has always been a difficult art. It is also taken for granted that geometry has its own intuition which makes geometry somehow more complicated to understanding as well as to dealing with. Still the world of geometrical ideas and problems is so challenging that it is worth to start even from really elementary things that are understandable for all.*

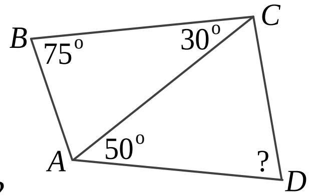
Key words: Geometrical intuition, challenging tasks, accessibility of problems, attractive formulations, development of concepts.

INTRODUCTION

Many years ago one of the authors was lucky enough to be present in the lecture on geometry given by the famous Latvian mathematician and educator Agnis Andžans. The author remembers pretty well that the lecture started with the advice that looking at the drawing one should start by asking simplest question concerning the essence of things.

For instance, looking at any segment is advised to ask whether in that picture you can find another segment of the same length. In the same way if you see any angle you are expected to look for another angle of the same magnitude. Similar questions of finding the congruent triangle or the triangle with not necessarily the same shape but with the same area might and should be raised as well. Despite the obvious nature and simplicity of all such formulations such approach appears to be highly effective.

For the illustration of what was told let us focus on the following task. Some angles in quadrilateral ABCD (see the figure) are known. If we assume that *BC = AD* then what is the



magnitude of the angle *ADC*?

(A) 30° (B) 50° (C) 55° D) 65° (E) 70°.

Applying our scheme we might start looking at the angle ABC and start asking whether in our picture one would be able to find another angle of the same magnitude. After that you will immediately be able to state with remarkable pleasure that this is indeed the case – because the angle BAC is of the same size. The standard argument of the sum of the angles in any triangle being 180° is the necessary base for that. Then continuing in the same spirit one would find the third segment which is as long as the segment *BC* is.

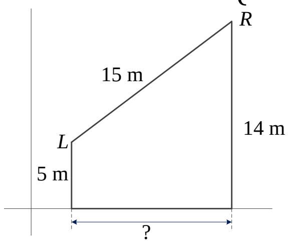
Many such similar and elementary tasks are very useful in the attempts to wake up our creativity that in the initial state starts from regarding the picture and simple undertaking something that might be of any use for the solution. Such problems might be and are met in the popular competitions of different levels. We are eager to remain at the accessible level. For that we will employ remarkably many problems which were posed in the

Kangaroo competition. This competition has accumulated a lot of problems through the years and is a real treasure for different areas, including geometry.

First examples

Examples 1, 2 and 3 is taken from Slovakian proposal for the Annual Kangaroo Meeting AD 2003 which took place in Paris. Paris might be called the native town of Kangaroo competitions since Kangaroo is born in France adopting Australian ideas for enrichment of popular problem solving.

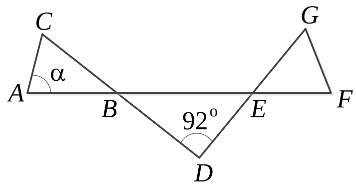
Example 1. There are two high bars prepared for the acrobats Leo and Raphael. Leo will practice 5 m above the ground, whilst Raphael will practice 14 m above the ground. There is a 15 m distance between the top ends of the bars (see picture). How long is the distance between the bottoms of the bars?



(A) 9 m (B) 12 m (C) 13 m (D) 15 m (E) 19 m

That problem is asking you to apply the Pythagoras theorem once.

Example 2. What is the size of the angle α in the picture if the triangle $\triangle ABC$ is isosceles with base AC and the triangle $\triangle EFG$ is equilateral?



(A) 92° (B) 76° (C) 56° (D) 28° (E) 20°.

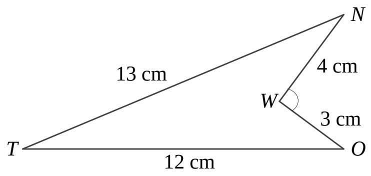
Remark

One might simply ask where the creativity of that problem is adding that from the computational point of view you are simply invited to do nothing but to find an angle α from the system of equations

$$\begin{cases} \gamma + \gamma + \gamma = 180^\circ \\ \gamma + \beta + 92^\circ = 180^\circ \\ \alpha + \alpha + \beta = 180^\circ. \end{cases}$$

The answer to that completely correct question might be that the creativity very often consists of some almost trivial steps, quite natural movements and remarkably simple attempts to look at the same matter from another, although often similar, point of view. Very often this stimulates and gives much more effect than it might be supposed when such an attempt is made. Simply speaking it is worth doing.

Example3. What is the area of the quadrilateral *TOWN* with the right angle at the vertex



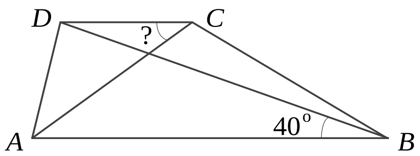
W as indicated in the picture?

(A) 20 cm^2 (B) 24 cm^2 (C) 30 cm^2 (D) 36 cm^2 (E) 48 cm^2

Let us mention that in the given case some creative element might be and is contained in an intuitive impulse or desire to end the concaveness or with non-convexity of the presented picture. For that it is enough to add one segment so completing not just one but even two right triangles in the regarded picture. After such completion the rest is done in a few seconds.

Starting from such simple or almost standard examples or initial patterns, sometimes called models, one can easily undertake something which wouldn't be so simple or at least a little bit more complicated. For instance, one might feel the wish to deal with some problems like the ones we are presenting for you below:

The trapezium *ABCD* with parallel sides *AB* and *CD* is famous for possessing the following properties: (A) The triangle *ABD* is isosceles with the base *AD* (B) The triangle *ACD* is isosceles with the base *AC* (C) The angle *ABD* has a magnitude of 40°. And the question is: what is the magnitude of the angle *ACD*?



(A) 30° (B) 35° (C) 40° (D) 70° (E) 75°

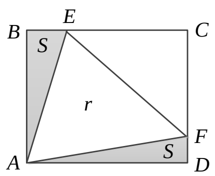
In the triangle *ABC* the points *K* and *M* on the side *AC* are marked in such a way that *M* bisects it. Moreover, it is given that *BM* = 3, *AK* = 1, *MC* = 2 and $\angle BMC$ = 120°. Prove that *AB* = *BK*.

The following two problems are taken from Latvian sources.

The convex pentagon ABCDE has three right angles $\angle A$, $\angle C$, $\angle E$. It is also known for sure that *AB* = *BC* and *CD* = *DE*. Find the $\angle ACE$.

Sometimes the proposed problems are not so difficult but still demand some technical abilities or computational skills but at the same time all such attempts give some impulse and wakes up our powers of geometrical imagination. This is something which also might bear the name of geometrical fantasy or even geometrical intuition.

In the picture below we see the square *ABCD* in which the equilateral triangle *AEF* is inscribed. Given that the two triangles have equal area *S*(*ABE*) = *S*(*ADF*) = *S* one is asked to compare *S* with the area of equilateral triangle *AEF*.

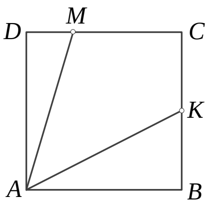


Five assertions for that are proposed and these 5 assertions are:

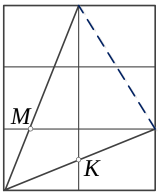
(A) $2S$ (B) $2\sqrt{2}S$ (C) $2\sqrt{3}S$ (D) $3S$ (E) $\sqrt{5}S$

Going beyond boundaries in geometry

Many times we hear about the usefulness of going beyond boundaries. This is understandable although it might be remarked that for going beyond boundaries one must know or feel what direction one has to choose.



Let us take another problem: points K and M are chosen on the sides BC and CD of the square ABCD in such a way that $BK = KC$, $CM = 2DM$ and one is asked to find the magnitude of the angle MAK.



From the point of view of trigonometry the problem is immediately converging to the really standard and common question: knowing the tangents of two angles, how to detect the tangent of their sum?

Still going the proper way beyond boundaries leads the problem to three simple application of really well known tool in geometry or to the theorem of Pythagoras.

Another more qualitative type of geometrical problems

This type of geometrical problems might be defined as the type of problems where you must think more about the properties of possible location of geometrical shapes rather than about quantitative relations connecting them. Such problems became especially valued in recent times when the knowledge of such geometrical facts is not so widely spread as we believe it used to be for years.

Let us present some examples of the kind. The first two of them are once again taken from the open Latvian Olympiad which is due to the efforts of already mentioned famous Latvian mathematician and educator Agnis Andžans:

1. Exactly 6 points are marked on the plane. Some of these points are connected with the line segments. It is known that each point belongs to at least 3 of such segments. Prove that from these segments one is always able to choose three segments ending in 6 different endpoints.

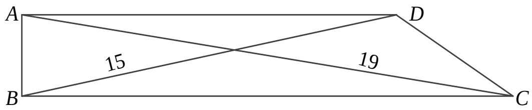
2a Draw 4 triangles in the plane such that each of these triangle has a common boundary with the remaining three triangles the length of which is at least 1cm.

2b What's about the possibility to draw 4 such quadrangles?

2c And what about the construction of 4 congruent hexagons with the vertices belonging to the grid of points with the integer coordinates and the sides parallel to that of coordinate axes?

3. The following problem is taken from the Sangakuland Olympiad sources which is from Japan.

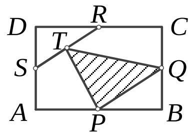
In the quadrilateral ABCD, $\angle A = \angle B = 90°, \angle C = 45°, AC = 19, BD = 15.$ Find an area of



ABCD.

The following two problems are also from the sort of problems which might be solved almost with one movement but only if you feel the direction of it.

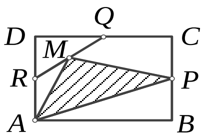
In a rectangle *ABCD* the points *P*, *Q*, *R* and *S* are the midpoints of the sides *AB*, *BC*, *CD* and *DA* respectively and let *T* is the midpoint of segment *RS*. What part of the area of the whole



ABCD is contained in the triangle *PQT*?

(A) 5/16 (B) 1/4 (C) 1/5 (D) 1/6 (E) 3/8

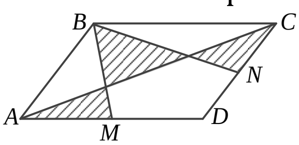
In a rectangle *ABCD* the points *P*, *Q*, *R* are the midpoints of the sides *BC*, *CD* and *DA*, respectively, and let *M* is the midpoint of segment *QR*. What part of the area of the whole



ABCD is contained in the triangle *APM*?

(B) 1/4 (B) 1/6 (C) 3/8 (D) 1/3 (E) 5/16

In the parallelogram *ABCD* points M and N are the midpoints of *AD* and *CD.* Assume that the area of that parallelogram is 1. What is the area of the shaded part of the picture?



Let us finish the section with one strikingly challenging and natural geometry problem taken from the Belarus math competition AD 1994 (Grade 10, problem 1).

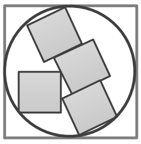
In the interior of the regular hexagon the point P is chosen in such a way that the sum of the distances from the point P to all the vertices is equal to its perimeter. Prove that in such a case point P coincides with the center of the hexagon.

Problems around the circle

Let us start with the following problem due to Belarus creative competitions AD 2002 in which cyclic quadrilateral ABCD is regarded. In that quadrilateral AB/BC = 3/4 and *ADC* is the right angle. Given that the area of quadrilateral *ABCD* is 44 determine its perimeter.

In the end let us enjoy some really creative, funny, not so difficult and extremely funny problems which we have found in the one of the last issues of Russian magazine Kvantik. They in turn found them on Twitter of Cathriona Shearer41.

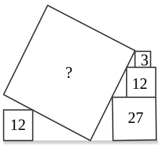
Let us enjoy some of them right now.



4 equal little shaded squares are located in the circle which, in turn, is inscribed into the big square. What part of the big square is shaded?

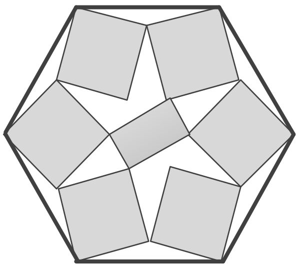
You would hardly believe that the most essential thing there seems to be applying the Pythagoras theorem once.

2. What is the area of the big square?



If you solved it successfully, did you really use all the given data?

3. Six congruent squares and one rectangle are inscribed into the regular hexagon as shown.



What is the relation of their common area to that of the whole hexagon?

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VISUALIZATION OF THE FIRST STEPS OF NUMBER THEORY FOR ELEMENTARY SCHOOL CHILDREN – A PYTHAGOREAN APPROACH

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Abstract: Influenced by the method of the Pythagoreans, who used pebbles to visualize numbers I have developed an approach to elementary math teaching where the students use coloured interlinking blocks and follow a few simple rules to visualize numbers, look for patterns, shapes and sequences, make their own mathematical creations and develop a sense of the more general principles of mathematics. Over the many years that I have been using the method in my math enrichment sessions at the Nueva School, a school for gifted children, I have found that this approach stimulates interest and enthusiasm for math, is a great motivator and can spark mathematical creativity, originality and a joy in the subject.

Keywords: Discovery Learning, Hands-on, Visualization of Number Theory, Figurate numbers.

INTRODUCTION

Number theory is the exploration of the properties of numbers. When we arrange numbers with manipulatives, the properties of numbers and the geometrical relations between them become visual. This aspect of numbers can naturally be accessed by an elementary school child irrespective of their previous mathematical experience or approach to numbers. It allows for an originality of creativity which a pure paper and pencil setting would not afford.

I teach students in small groups of up to 4, differentiated through grades 1-4, in one hour math enrichment sessions outside the classroom setting. Some students I see regularly throughout the school year, others for only a few sessions. The composition of the groups can change on a weekly basis.

Individual students vary across a wide spectrum of math experience and approach to math, across the grades and within them. Some students wait for instructions, start from the beginning and build systematically step by step. Other students branch out very quickly, bursting with ideas. To discover the math behind helps them to navigate their creativity.

Even for those students fascinated by computation and large numbers, and who are far beyond their grade in speed, skill and proficiency in calculation, hands-on math has a role to play in opening their eyes to more general concepts. This in turn supports their arithmetical skills when dealing with much larger numbers, for example Pythagorean triples (whole number solutions for the equation a^2+b^2 = c^2 like the familiar 3^2+4^2 = 5^2) or writing a formula for the summation of the square numbers or triangle numbers.

ROLE OF THE TEACHER

I teach a mindset for inquiry, of asking probing questions, trying out ideas, finding out if they work and, if so, experimenting to find out why they work. For higher grades this can be translated into more formal mathematical/algebraic language. My role is to encourage, support and stimulate students to think mathematically by providing an environment in which students are actively encouraged to discover patterns and regularity, and to ask questions and find their own ways of answering them. I aim to draw out students' creative and ingenious ideas by thoughtfully guiding them to their own discoveries and insights into some aspect of math to which they can claim ownership.

PRACTICE

When meeting a set of younger students for the first time I might encourage them to take a handful of counters, place them in a pile on the table in front of them and estimate how many counters there are in the pile. Is there a better way to estimate the number? Perhaps spread them out in a geometrical pattern in a shape, which will help not only to estimate but also to determine exactly the number of counters. Why does the shape help? If they make a rectangle or a square of counters, they can count the rows or columns rather than counting each counter individually. It is a shortcut. It provides insight, turning addition into multiplication. In enlarging the squares to the next bigger size, the students encounter the L shaped gnomon.

First grade students, following the Pythagoreans' first principle of number can be encouraged to arrange odd and even numbers into clearly identifiable representative geometrical shapes using the blocks. They may make line numbers and arrange them into triangle numbers. In manipulating the numbers, putting them together and breaking them apart, the students gain an intuitive feeling for how the shapes go together

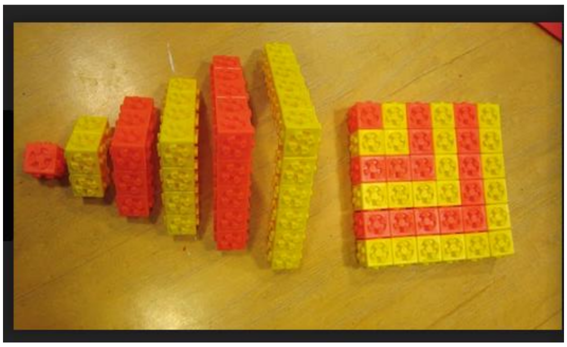
Geometrical shapes go hand in hand with numbers. That a square number, by which we mean a number multiplied by itself, when made with blocks actually looks like a square can come as a surprise. A triangle number looks like a triangle. Following simple rules which guarantee order, harmony and regularity (start off at the smallest size; go up step by step in sequential order; each term in the sequence must be a visual replica of the previous term only bigger; when rearranging the designs use the same number of blocks) the students can discover that by adding consecutive odd numbers starting from 1 they can produce square numbers. By adding consecutive even numbers, starting from 2 they can produce oblong numbers. By adding consecutive odd and even numbers, starting from 1 they can produce triangle numbers, and by combining 2 identical triangle numbers they get oblong numbers. When they combine 2 consecutive triangle numbers they will get squares. The various possibilities seem to be surprising and yet predictable at every iteration.

By experimenting with shapes, students get to know different kinds of numbers, their properties and relations, concepts of arithmetical and geometrical sequences and their summations. Square and triangle numbers become essential building blocks for other polygonal and polyhedral numbers. Seeing unexpected relations between two different-looking aspects of the same thing can enable older students to establish algebraic equations between those aspects.

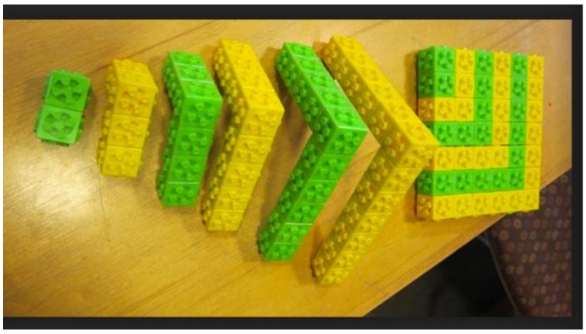
For higher grades, discoveries made in two dimensions can play out in three dimensions, often with intriguing math. The resulting polyhedral numbers can form fascinating crystal-like structures, which can morph into cubes, squares and lines to visualize the polynomials involved. Students can discover that octahedral numbers are the sums of two consecutive square-based pyramid numbers. Each square-based pyramid number is the sum of two consecutive triangle-based pyramid numbers (tetrahedral numbers) reminiscent of the square number in two dimensions made from joining two consecutive triangle numbers.

Students sometimes ask, when making tetrahedral numbers, how one can predict the number of blocks in the structure. If they know how the triangle numbers go, they simply add the triangle numbers. A pyramid of 4 layers means they have to add the first four triangle numbers. But, how would they determine the number of blocks consumed by a pyramid 20 layers high, or 100 layers high? With other related arrangements they can discover a formula for the nth term of the pyramid. If it is n layers high students can determine the number of blocks used. There are different ways of approaching this both visually and numerically. The visual hands-on approach is a stepping stone to algebra with natural numbers which we express with n, and which is accessible for some 3rd graders and above. For younger grades we can omit the algebra while nevertheless paving the way towards it.

EXAMPLES OF NUMBERS VISUALIZED BY GEOMETRICAL SHAPES



The first 6 consecutive odd numbers represented by red and yellow L shaped stripes or gnomons. What the gnomons are telling us here is that when all the numbers are added together they will form a perfect 6 x 6 square. The visual pattern suggests it can be generalized and turned into a rule: $1+3+5+ \ldots 2n-1=n^2$



Visualization of the sum of the first consecutive even numbers starting from 2. When put together the even numbers form an oblong rectangle (an oblong number) $2+4+6+\ldots+2n=n(n+1)$ where n = 1,2,3… Here the oblong rectangle has a long side one block longer than its short side

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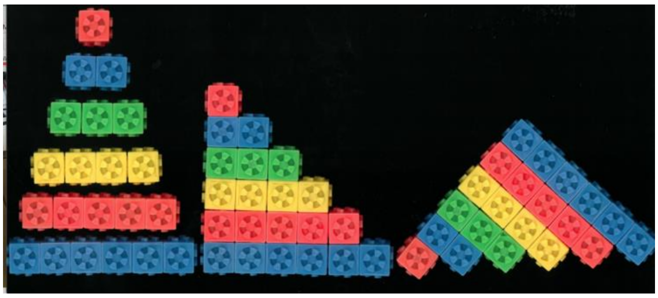
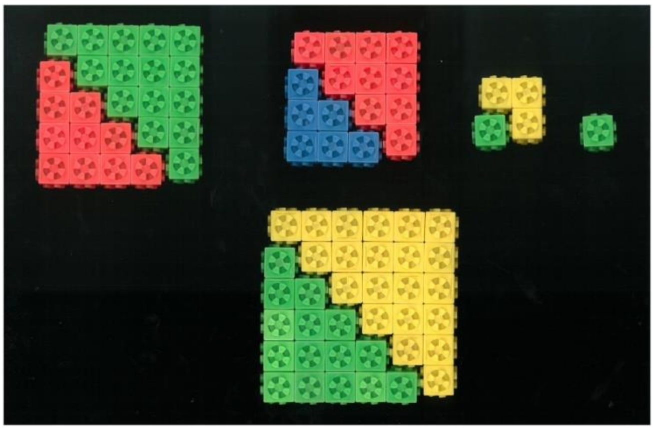
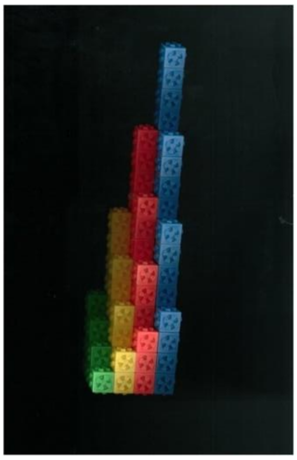


Line numbers. Every number can be visualized with blocks as a straight line. Here are the counting numbers and their sums. On the left are the first 6 line numbers. Joined together in a shape similar to a triangle they represent the sum of the first 6 counting numbers 1+2+3+4+5+6 = 21. Line numbers can be joined together to form triangle numbers.

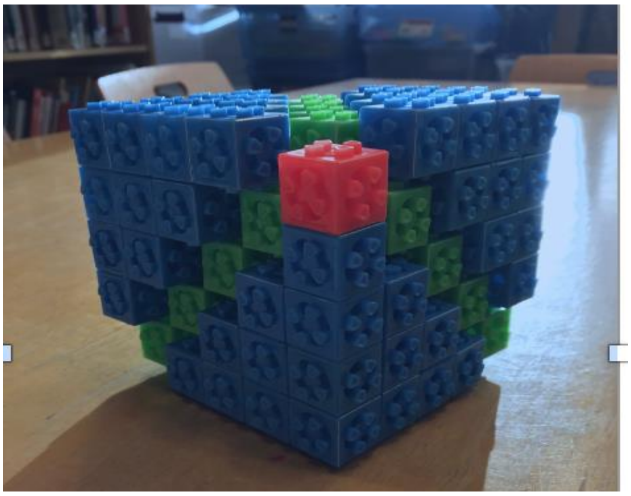


The sum of two consecutive triangle numbers seems to suggest they always make a square number T(n)+T(n+1)=(n+1)^2



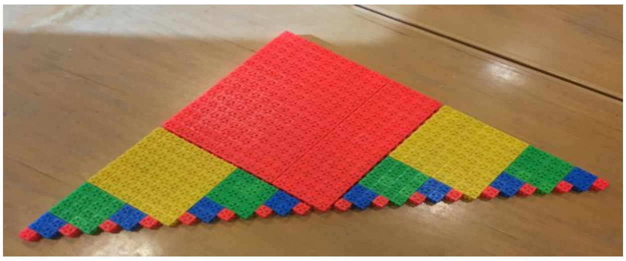
The entries of a multiplication square as a summation of the multiples of the triangle numbers.

Growing crystal-like structures from shells of triangle numbers.



The crystal-like structure lies hidden inside the blue walls of a perfect cube.

Interesting hollow cubes. Students create such structures naturally and spontaneously. The polynomials would capture and challenge students in the higher grades.



The elegance of the perfect number 496 – a triangle number.

CONCLUSION

There are so many discoveries to be made and a richness and depth to the approach for students and teachers alike. Steps towards the theorem by Pythagoras can be encountered, and shown visually. With counting and patterns, students experience the blend of arithmetic and geometry, which drives mathematical discovery; it has driven the subject for millennia. It inspires students to ask questions, guess, speculate and hypothesize. Stimulating the students' own questions is even better, and motivates them to solve their own problems.

If the effectiveness of the visual hands-on method can be gauged by a student's motivation, then the enthusiasm and pleasure of doing math, which I observe on a daily basis is a rewarding validation, which has been confirmed to me by parents and students many times over.

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INCUBATING MATHEMATICAL CREATIVITY THROUGH A MOLECULAR GASTRONOMY 101 SATURDAY ENRICHMENT CAMP

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***Abstract**: This project report addresses creative applied mathematics used outside the regular education classroom. It examines universal themes and standards of mathematics through the pedagogical framework of the Torrance Incubation Model of Creative Teaching and Learning (TIM) during a university-sponsored Saturday Enrichment Camp for gifted and talented learners. Enrichment camp instructors included experienced gifted facilitators, parents of gifted children, and grown-up gifted children who volunteered their time, energy, culinary expertise, and specialized equipment to explore three modern culinary techniques in a three-hour molecular gastronomy class for gifted and talented learners ages 8-14 years. The hands-on Saturday Enrichment Camp with real world application required participants to apply mathematics and technology to prepare a three-course meal using seasonal locally sourced farm food ingredients from a rural Midwestern community in the United States.*

Key words: Creativity, giftedness, mathematics, enrichment, Torrance, molecular gastronomy

MATHEMATICAL CREATIVITY AND GIFTEDNESS

Young learners identified with gifts and talents typically receive specialized instruction for core academic subjects including reading and math in public and private schools. Qualified school personnel assess gifted learner potential in these subjects using a variety of standardized instruments to determine academic potential and need for modified curriculum. When placed in gifted services, students then receive individualized instructional plans designed to address their exceptionalities. Content standards in academic subjects determine expected outcomes, and individuals with gifts and talents may function several grade levels above their chronological age ability. In mathematics, for example, identified gifted learners may receive enriched and accelerated instruction above grade level so they continue to learn and develop advanced potential.

Hoeflinger (1998) found identification of mathematically precocious students an elusive process, as they may underachieve in the regular education classroom, show little interest or effort during math instruction, or score below their potential on achievement tests. Moreover, when classroom teachers teach only basic formula and rules, mathematically gifted learners miss opportunities to demonstrate their creative thinking processes. A common characteristic of mathematically gifted children relates to their ability to grasp multiple layers of complex problem-solving and suggest a variety of solutions or strategies to solve a problem. Students with academic giftedness in mathematics view the world through a mathematical lens, interpret scenarios embedded with mathematical language, create graphs and tables to solve problems, and use manipulatives to explain processes. They ask *why* and conduct their own investigations to prove their point (pp. 244, 246).

Kozlowski, Chamberlin, and Mann (2019) reviewed characteristics of mathematical giftedness and operationalized mathematical creativity as a psychological construct blending fluency, flexibility, originality, and elaboration as indicators (p. 506). They examined mathematics and creativity within the Wallas Gestalt Model of Creativity with its four stages of preparation, incubation, illumination, and verification with their respective associations between conscious and unconscious processes. Despite the complexity of defining domain-specific mathematical creativity and its unconscious processes, they found considerable research addressed quantifying creative thinking abilities. Kozlowski et al. (2019) advocated research to examine the relationship between teaching methods and the role of student affect toward mathematics in the classroom.

TORRANCE INCUBATION MODEL OF TEACHING

In 1996, E. Paul Torrance developed the Incubation Model of Teaching (TIM) with its three interactive, sequential stages to support creativity in the classroom (Hébert, Cramond, & Neumeister, 2002). The model supported teachers during lesson planning, delivery, and extension with a menu of instructional strategies for each of the three stages. Stage One, *Heighten Anticipation*, served as a warm-up with six functions in lesson planning; Stage Two, *Deepen Expectations*, embedded problem-solving processes with eight strategies in lesson delivery; Stage Three, *Keeping it Going*, extended learning with real world applications with five metaphors to inspire creativity to extend the lesson. This study proposes the TIM to bridge the gap between psychological constructs of creativity, instructional strategies, and student affect during a range of learning experiences outside the classroom. School budget reductions in programming, less school staff to deliver modified services for high ability learners, and concerned parents seeking talent development opportunities for their children increase the need for out-of-school enrichment experiences. This study examines mathematical standards, concepts, relationships, and applications during the Saturday Enrichment Camp through the pedagogical framework of the TIM. Similar to the grassroots *makerspace* movement of "hands-on making, creating, designing, and innovating" (Peppler & Bender, 2013, p. 23), the Molecular Gastronomy 101 enrichment camp blended community involvement, high interest, and creativity through a range of activities and variety of assembly materials.

SATURDAY ENRICHMENT CAMPS

Saturday Enrichment Camps for gifted and talented children provide out-of-school experiences to explore high interest areas with real world applications as a group in a supportive environment. Infused with higher-level thinking and creative problem-solving skills, enrichment camps provide a single-focus academic or creative experience designed to stimulate their intellectual curiosity for further exploration after the camp. Although school districts, universities, academies, and museums often sponsor enrichment camps, various organizations, businesses, and community groups also possess facilities and expertise to offer enrichment programs. Saturday enrichment programs include a series of camps on consecutive Saturdays during an academic semester during a morning or afternoon camp of several hours duration with multiple sessions within each camp. Camp programs organize sessions by clustered ages or grade levels sessions conducive to common interests and cognitive ability. This environment provides hands-on engagement essential to support social, emotional, and cognitive growth in a non-competitive environment without homework or grades.

Enrichment programs provide excellent opportunities otherwise unavailable in schools for diverse gifted learners to learn and practice academic *thinking dispositions* needed for success in life and as career explorations at young ages. In mathematics, thinking dispositions such as *perseverance*, *thinking and communicating with clarity and precision*, *thinking flexibly*, *thinking interdependently*, and *remaining open to continuous learning* found in the Habits of Mind (Costa & Kallick, 2008) prove useful. Broad-based thinking dispositions relate well to more specific discipline-based ways of thinking and behaving. For example, the National Council of Teachers of Mathematics (NCTM) Evaluation Standard 10 Mathematical Disposition advocates confidence, flexibility, perseverance, inventiveness, reflection, valuing, and appreciation of mathematics as both a tool and a language (NCTM, 1989, 2000). Moreover, mathematics content standards support universal themes and generalizations for change, conflict, exploration, force or influence, order, patterns, power, structure, systems, and relationships (Kaplan & Curry, 1985). Combined with innovative culinary techniques, mathematical standards and themes required creative planning, careful preparation, and evaluation of practices. For example, community supported agriculture (CSA) flexibly delivers seasonal food products despite unpredictable weather conditions that affect crop growth and harvest. Some molecular gastronomy techniques depend on lightweight food substances that require digital scales to measure ratio-based amounts accurately. The outcome of a three-course molecular gastronomy meal calibrates technology correctly to cook and hold foods at precise temperatures.

MOLECULAR GASTRONOMY 101

Molecular gastronomy represents a type of modern cuisine based on distinct flavor sensations created during physical and chemical transformations of manipulated foods. In the late 1980s, French agricultural scientist Hervé This experimented with thousands of transformations in molecular gastronomy designated as the *scientific study of cooking*. A classic study examined the impact of temperature on eggs using *sous vide* or "under water" technique with a low temperature immersion circulator. Other molecular gastronomy techniques include *spherification* of substances for a caviar effect, creating foams with a whipping siphon or immersion blender, flash freezing with liquid nitrogen to make ice cream or to shatter fruit, food dehydration, powderization with maltodextrin, and infrared sensors to monitor cooking times. This experimentation provided the foundation that transformed culinary arts used in high-end restaurants throughout the world.

Michelin three-star Chef Ferran Adrià of the former elBulli in Spain manipulated food flavors to infuse rose scent into mozzarella cheese, combine melon and ham, and create pine nut marshmallows. Chef Grant Achatz of the three-starred Alinea in Chicago blended fruit flavors and inverted sugar to create his iconic edible helium-blown balloon signature dessert. Former Chef Homaro Cantu of Moto restaurant in Chicago served edible paper menus constructed of corn and soy, laser cooked fish, and used the West African *miracle berry* to sweeten sour foods. Chef Cantu also transformed crabapples, cactus, and hay ingredients from his backyard into barbecued steak. These chefs achieved game-changing innovations in the culinary world based on investigations in science labs. World-class chefs represent celebrity status, and the plethora of cooking shows on television attest to the appeal of culinary arts for children as well as adults. Moreover, the culinary market now provides affordable equipment and products needed for molecular gastronomy cooking techniques for the home chef. These culinary advancements demonstrated the universal themes of *change* and *growth* in mathematics (Kaplan & Curry, 1985). The Saturday Enrichment Camp challenged campers to use applied mathematics creatively and innovatively as they encountered new technology and ratio-based “recipes.”

Stage 1 Heightening anticipation.

Months of behind the scenes preparations set the stage for the Saturday Enrichment Camp. The warm-up phase included a variety of physical, mental, and social activities designed to *get attention* for a “problem” to solve and stimulate curiosity about the subject. The “problem” targeted out-of-school enrichment experiences, and a group of gifted facilitators from a local school district met with the Director of Gifted Education at a nearby university to organize its first Saturday Enrichment Camp, “Molecular Gastronomy 101.” Considering school events and holiday breaks, they selected a Saturday in late April, multiple grade levels three through eight, affordable cost, convenient location at the university without charge, camp curriculum, and duration of three hours. The gifted facilitators notified parents of gifted students at in their respective schools, and the Director of Gifted Education announced the enrichment camp during a Faculty Senate meeting. A university graduate assistant designed a camp flyer with essential information and created a booklet including the camp schedule, class content, vocabulary, and camp instructor biographies. As an innovative camp experience, prior knowledge and skills only required the ability to use standard measuring spoons and cups, read digitally displayed information, follow a series of sequential steps in a procedure, and understand the concept of ratios.

As an initial effort, the Saturday Enrichment Camp included volunteers to keep costs affordable yet provide funds for future enrichment camps. For example, a retired Gifted Coordinator who co-founded a Summer Enrichment Camp in a large school district served as registrar. A grown up gifted Honors College student took photographs. Several school-based personnel assisted with small group rotations through three molecular gastronomy techniques. A parent of gifted children with expertise in molecular gastronomy led small group explorations. A professional chief who taught a molecular gastronomy class at a regional culinary center mentored the project from start to finish. A parent donated chicken, corn, and blackberry juice products from her local farm, and local gifted facilitators assisted with food preparation. The camp director secured college grant funding to purchase farm vegetables and greens from a local farmer and maltodextrin, and agar from Amazon.com.

The Teachers College provided classrooms at no charge, and the camp staff stored perishable dairy products in the faculty lounge refrigerator. Without traditional kitchen equipment, they secured a former science lab classroom with a deep sink to vacuum “hand press” plastic bags under water for sous vide and used its extended countertop to prepare and cook foods. The Saturday Enrichment Camp delivered an intensive experience infused with applied mathematics experienced creatively throughout the program. The pre-camp experience identified and motivated potential campers, their parents, teachers, and camp staff by marking their calendars for the Saturday Enrichment Camp. The coordination of activities with parts that worked together and involved subsystems related to the universal mathematical theme of *systems* (Kaplan & Curry,

1985) with campers learning in whole groups, applying skills in small groups, and evaluating progress individually.

Mathematical dispositions (NCTM, 1989, 2000) form a significant element in assessing giftedness and creativity. The Molecular Gastronomy 101 enrichment camp provided an out-of-school experience for camp instructors and staff to observe approximately 15 high ability students in whole group learning and hands-on small group exploration settings. Campers demonstrated mathematical dispositions to solve problems and communicate ideas; flexibly explore mathematical ideas and try alternative solutions; persist in tasks requiring precise mathematics, approach mathematical tasks with curiosity; reflect on their own performance; value mathematical applications in other disciplines; and appreciate of mathematics in their own culture. For example, campers explored ratios by quantity and quality of food ingredients evaluated by taste, color, and size in a meal course and recorded observations in their camp booklets. They assessed timing required for droplets of juice into containers of chilled vegetable oil to form encapsulated spheres at the desired size and shape. They reflected these dispositions through verbal and written comments in booklets.

Stage 2 Deepening expectations.

The Molecular Gastronomy 101 enrichment camp consisted of two parts: whole group community supported agriculture (CSA) demonstration and small group exploration of three molecular gastronomy techniques. During the whole group demonstration, a local farmer discussed the farm to fork movement and CSA as a system with shareholders supporting a small farm operation. The presentation included a demonstration of microgreens as a simple way to grow vegetables for any meal. The farmer guided campers as they planted sprouted sunflower seeds and cut microgreens for the salad course in the three-course luncheon. When campers planted small sprouted seeds at the correct depth for optimal growth, they observed the relationship between size, texture, and shape from the fully-grown microgreens they washed and dried for the salad starter course.

The whole group demonstration also introduced three molecular gastronomy techniques used in the three-course luncheon: sous vide, spherification, powderization. *Sous vide* cooks food in a vacuum-sealed plastic bag at a slow constant temperature. Sous vide requires an immersion circulator to heat and circulate water to a precise and constant temperature. *Powderization* transforms food textures with maltodextrin and a high-fat liquid like olive oil or Nutella spread into a flavorful powder. *N-Zorbit* contains *maltose*, a food additive similar to starch. Spherification encapsulates liquid food substances such as juices or oils into spheres using *Super Agar,* a form of agar agar derived from red sea vegetables used to thicken foods. Campers experienced the universal mathematical theme of *relationships* during the whole group farm to fork and molecular gastronomy demonstrations through *purposefulness* (Kaplan & Curry, 1985) as they prepared microgreens for the salad starter course and gained knowledge needed to plan the main and dessert courses. They understood these innovative culinary practices required mathematics applied creatively.

The whole group "deepening expectations" phase presented mathematical problems of planning, securing, and purchasing ingredients for a three-course meal. They estimated the element of time needed for innovative techniques such as sous vide as well as freezing and thawing food items. During small group hands-on explorations, campers determined preparation time by verifying food temperatures during the spherification technique requiring cold vegetable oil to encapsulate liquid substances. They understood the relationship of cooking time and temperature needed to cook chicken and eggs safely to a precise temperature using sous vide and a non-touch infrared digital thermometer.

Stage 3 Keeping it going.

After the farm to fork whole group demonstration, campers *extended learning* through hands-on small group explorations culminating in a high-class three-course culinary experience. Campers formed small groups in the former science lab room, wore aprons, practiced knife skills under supervision, selected flavor profile ingredients, and rotated through exploration stations, thus experiencing applied mathematics creatively while learning molecular gastronomy techniques. A bonus technique featured a whipping siphon using nitrous oxide to quick freeze ice cream for dessert. Students used modern culinary experiences to assemble the three-course meal with a microgreen salad starter dressed with spherified olive oil and blackberry juice pearls, sous vide cook the main course chicken with lemon slices and fresh herbs served with spring vegetables and corn side dishes, and a frozen ice cream dessert topped with powderized Nutella spread.

Campers used multiple senses to create flavor profiles with fresh lemon, garlic, rosemary, thyme, butter, and olive oil, as well as salt and pepper, for sous vide chicken. They selected, crushed, and chopped seasonings by size, shape, and color to provide a pleasing blend of flavors. They released droplets of blackberry juice and olive from a plastic squeeze bottle into vegetable oil chilled in tall cylindrical glass jars transformed into salad dressing "pearls." They evaluated the taste and texture of tender sous vide chicken, corn, and spring turnips. Throughout the hands-on small group rotation, campers observed and practiced *mis en place* with culinary utensils and equipment ready for use and removal as a demonstration of the universal mathematical theme of *order* (Kaplan & Curry, 1985).

Mathematical applications in the hands-on exploration groups included measuring the lightweight substances of N-Zorbit (maltodextrin) required for powderization and Super Agar used during spherification. Campers used food ratios, rather than standard recipes, to transform food flavors into new shapes and textures. They relied on their sense of taste and smell to season sous vide chicken in vacuum compressed and sealed plastic bags. Sous vide immersion circulators cook and hold chicken and eggs at precise and constant temperatures needed for food safety. Campers observed infrared digital thermometer readings to check doneness of sous vide chicken and eggs in the water bath containers.

CONCLUSION

The Saturday Enrichment Camp "Junior Chefs" used individual camp booklets to take notes, record impressions, and research culinary references for further study. For example, they could further explore fantastic applications of molecular gastronomy practiced at three-star Michelin restaurants or enjoy every day applications to cook eggs precisely with their enrichment camp "recipes" at home. Upon completion of the Saturday Enrichment Camp, Junior Chefs reflected upon the benefits of Community Supported Agriculture and impact of the farm to fork movement on their local community. They spoke to parents and siblings of "the best chicken they ever ate," slow cooked without traditional kitchen methods requiring a stovetop or oven. Junior Chefs received digital badges through email acknowledging their achievement preparing a three-course meal using molecular gastronomy techniques, demonstrating team-building skills, and building individual flavor profiles. Based on evaluations, the Saturday Enrichment Camp leadership team planned Crime Scene Investigator (CSI) and Entrepreneurship enrichment camp experiences in subsequent years. The three-stage TIM pedagogical framework provided a richly layered creative out-of-schools learning experience that generated multiple opportunities for Junior Chefs to view, examine, and assess culinary facts, concepts, and principles as creative and gifted mathematicians. Junior Chefs solved reality-based mathematical questions in order to cook a high-class three-course meal. As a result, they experienced applications of mathematics in areas where the use of mathematical methods might seem less obvious. They gained an impression about the influence of mathematics on an activity such as cooking that might superficially seem more creative rather than strictly logical. Through a variety of mathematical tasks used for a range of culinary activities, Junior Chefs observed a creative usage of mathematics as they investigated knowledge and skills needed to perform the range of culinary tasks. As a specialized type of community-based makerspace, the project demonstrates real-world learning outside the classroom as Junior Chefs practiced applied mathematics creatively, collaboratively, and communicatively.

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IT'S NOT ALWYS SIMPLER TO USE "MAKE IT SIMPLER"

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Abstract. *This manuscript describes the problem-solving strategy of "Make It Simpler" in a context that goes beyond using more convenient numbers to exploring metaphor and content change. Challenging problems that do nt appear to have clear or readily identifiable pathways to solution are used to illustrate how "make it simpler" through the use of metaphor and/or context shifts can lead to increased discourse and deeper comprehension of underlying concepts.*

Key words: *Problem solving; connections; reasoning; quantitative literacy; metaphor; isomorphism*

INTRODUCTION

Among the strategies identified as useful tools to use in solving problems are, for example, drawing a picture, making a table, looking for a pattern, or acting it out. A strategy often referred to as Make it Simpler, is frequently illustrated by inserting "easier" numbers in a problem while at the same time eliminating extraneous information. Although this method works in certain situations, there may be a better (simpler?) way.

For example, consider the following Milk-Syrup Problem.

> There are two containers, one with milk, and the other with chocolate syrup. A certain amount of syrup is poured into the milk and then the mixture is fully stirred. Next, the same amount of the mixture of the "chocolate milk" is poured back into the syrup. The question: Is there more syrup in the milk container, more milk in the syrup container, or the same amount of milk and syrup in each container?

Often, the response is that there is now more syrup in the milk container. This answer is typically justified by an explanation that "pure syrup" was poured in, but that mixed syrup and milk was poured back. A second common answer is that there is no way to tell since we don't know the size of the containers – which, it turns out, is not a relevant factor.

Yellow Beads and Blue Beads

Changing the context, while maintaining relationships in the problem, may help alleviate certain confounding variables. In this case, the realization that syrup is not milk and milk is not syrup. Instead of containers of milk and syrup, use two jars, one with 1000 yellow beads and the other with 300 blue beads. Here, it is clear that yellow beads are not blue and blue beads are not yellow. If one carries out the action described in the Milk-Syrup problem, it will become clear that placing X (some number) blue beads in the yellow beads jar and mix, followed by taking a random X (same number) of beads back to the blue jar will result in the same number of blue in the yellow jar as there are yellow in the blue jar. A number of trials, changing the value of X each time, will provide convincing evidence that since X beads "went over" and X beads "came back," then for every blue bead that did not "come back," a yellow bead must have taken its place.

Boys and Girls Bus Problem

Taking this additional step, consider two buses full of children, one with all boys and the other with all girls. In this case, we will allow for different size buses (as we could have for the syrup/milk and the blue/yellow versions), but now we limit the capacity of the containers – the buses will be full so no extra seats available. The reason for this adjustment will become obvious in the problem set up.

> Two buses on a field trip stop for a break. Bus #1 is filled with girls and Bus #2 is filled with boys. X number of boys get off the Boys' bus and board the Girls' bus. When the driver is ready to leave, he notices that not everyone on the bus has a seat, so he tells X extra passengers to go to the other bus. X students get off – some boys and some girls, and board the other bus. Are there now more girls on the Boys' bus, or more boys on the Girls' bus, or are there the same number of boys and girls on each bus?

This bus version of the problem may be modeled by using red/black playing cards, two different colored chips, or heads/tails of coins. Begin by making two piles, one red and the other black (or two separate groups of colored chips or two piles of coins, one pile Heads up and the other Tails up). Move X (some number) from the first pile to the second pile. Mix them up.

Move X (same number) back from the second pile to the first. You will now see, for example, 5 red cards were moved over and 5 any color cards were moved back. If it happened that only 2 red cards were moved back, then 3 black cards must have come with them to make a total of 5 (while 3 red cards and 2 black cards left behind). Regardless of what the actual number are that are moved, it will always result in the same number of "odd" colored cards in each black and red pile.

This version of the Make it Simpler strategy accomplishes so much more than simply changing numbers. It changes the focus from making the arithmetic easier, to focus on the discourse of reasoning rational thought, as well as to finding connections between models and contexts.

Map Temperature Problem

Another version of Make it Simpler can be found by building your way up to the level of sophistication required for solution. Frequently, this application involves stating a problem AND the answer to the problem, but then asking, "How do we know?"

Consider the following:

> Given a map of any region with a circle drawn on the map (of any radius). There will always be two points on the opposite ends of a diameter that will have the same temperature. How do we know?

Take a moment and think about the problem and its implications. The problem does not limit the size of the circle. It also does not ask you to actually find the two points – it simply assures you that at any given point in time, there will be two such points. The real problem is not in finding the two points, but instead, finding a way to explain how we know there are ALWAYS will be such points.

How can we go about tackling this problem? To set a plan of attack on this map problem, let's consider a simpler, perhaps seemingly unrelated, problem – specifically, a rate/time/distance query that will lead us toward a way of thinking about continuity, the cornerstone of one solution to the posed problem.

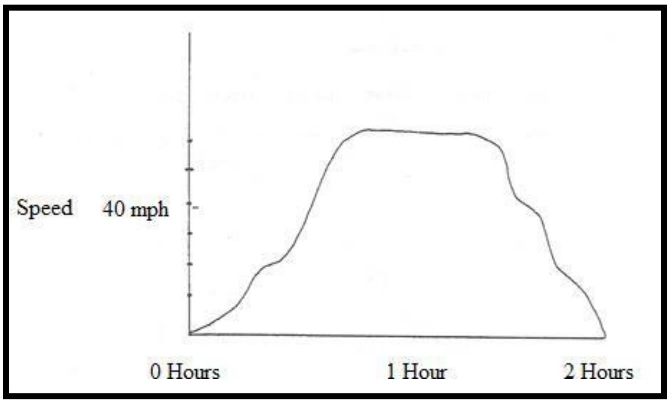
The Car Travel Problem

This car trip problem has a vague connection to the temperature problem in that both seem to be asking for an existence "proof" of a proposed conclusion. If one can find a path to solving one problem, it may help in solving the other.

> A family began a car trip at 2:00 PM. After driving for 2 hours, the car traveled a total of 80 miles. Was there ever a time when the car's speedometer showed "exactly" 40 miles per hour?

Aside from both problems calling for a justification, what other potential analogies exist between these two problems? If one considers the starting point as one endpoint (A) of a line segment and the point 2 hours later as the second endpoint (B), then to get from A to B you must have traveled 80 miles in 2 hours thereby averaging 40 miles per hour. Of course, you started at zero miles an hour so you didn't travel 40 mph for the entire trip – there must have been at least one period on the trip where you travelled faster than 40mph to make up for the time that you spent driving under 40 mph. Consequently, there are at least two points between A and B where the speedometer reads exactly 40 mph. (Bonus: MIGHT there be three such points, or does the number of such points have to be an even number?)

A graphical representation of the car travel problem might help us make the next connection in our journey to solve the temperature problem. See Graph 1



Graph 1: Changes in speed over time on a trip

We don't have the specific details of the speeds at any given point in time except, at the start the car is "at rest" and begins increasing speed to a point beyond 40 mph at some point. Also, we know that the car's speed will have to revert back to 0 mph at the end of the trip. Consequently, the graph shows a possible record of speed and distance, travelling from point A to point B. If the travelling line stayed below the horizontal 40 mph for this trip, it would not be possible to average a speed of 40 mph. Therefore, there MUST be at least two points during this trip where the speedometer reads "40 miles per hour."

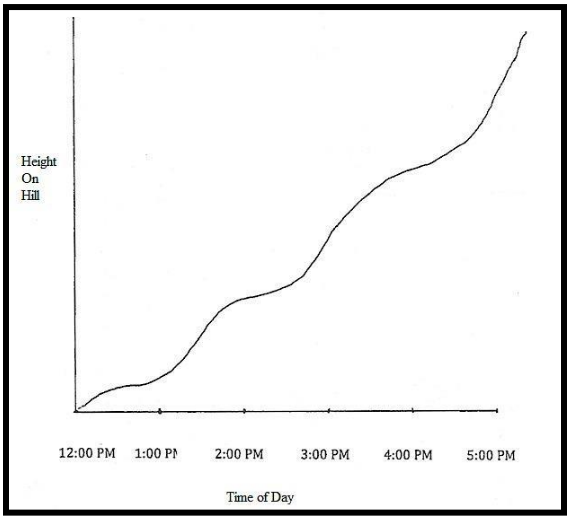
Hiking Problem

Now, consider another variation on this theme that involves a trip with a starting point, a stopping point, and a measurable attribute, such as time or speed.

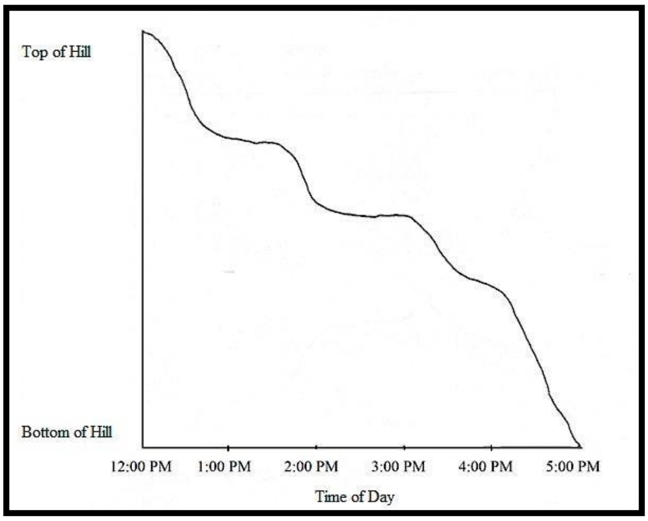
> At noon, a hiker began to climb a hill following a single path. She arrived at the top of the hill at 5pm. After spending the night at the top, at noon the next day, the hiker began the trip down following the same path, and arrived at the bottom at 5pm. After a brief moment of reflection, the hiker realized that there was at least one place on the way down where she was at exactly the same place at the same time the day before. How does she know?

In this problem, a coordinate graph with the horizontal axis denoting time and the vertical axis showing the height of the hill, can be used to chart possible trips the hiker could take, starting at the bottom of the hill on Day 1, and starting at the top of the hill on Day 2. As with the Car Travel Problem, we do not know the actual events of the hike other than the beginning time, the ending time, and the trail followed. We DO NOT have any information on rest stops, running, snack breaks, etc.

A graphical representation of the Hiking Problem (similar to the one used to interpret the Car Travel Problem) might again help us make the next connection to solve the Temperature Problem. Graph 2 shows a potential pathway for the hike up (the curve starting at the bottom of the hill at noon)

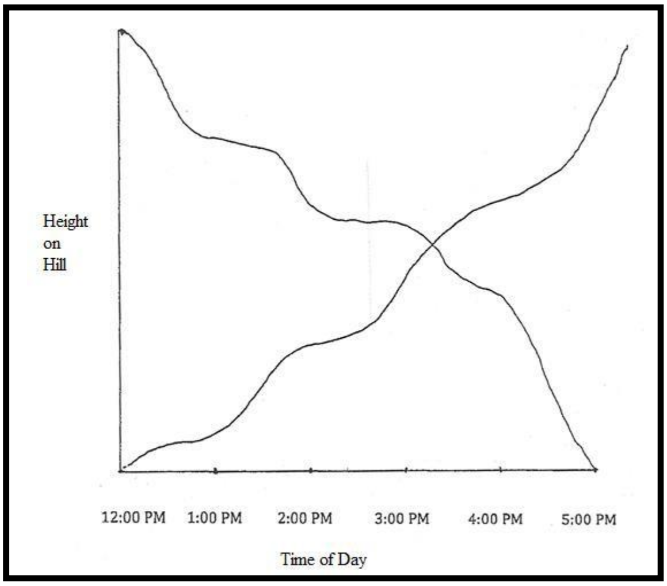


Graph 2: Possible Hike Up the Hill



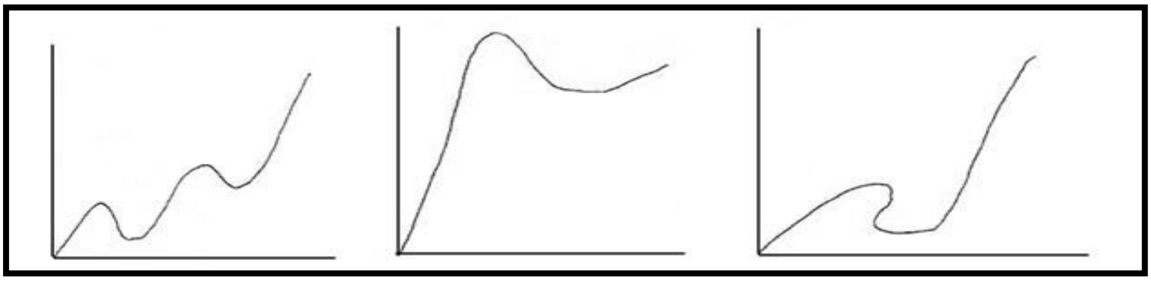
Graph 3: Possible Hike Down the Hill

Although Graph 3 shows a potential pathway for the hike down (the curve starting at the top of the hill at 12 pm), we can see that each trip (hiking up and hiking down) that took 5 hours to complete and, when you overlap the two days, the graphs must intersect (at least once) during those 5 hours. The intersection point represents the hiker at the same time of day at the same point on the trail. Could the intersection occur more than once?



Graph 4: Combing Possible “Up” and Down”

As an aside, it is also interesting to consider the feasibility alternative “hiking up” graphs.



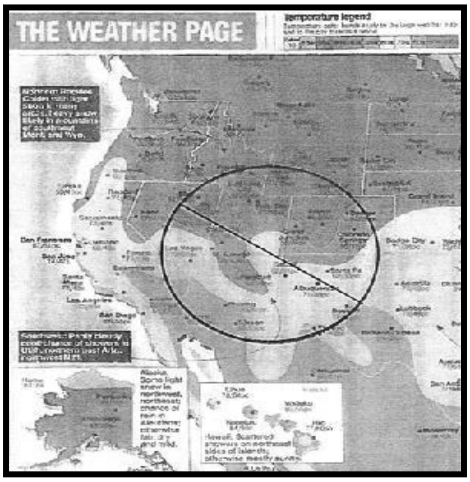
Graph 5: Other Possible Hikes Up the Hill

Another Look: Map Temperature Problem

Graph 6: Temperature Map of Western US.

Fact (??) - Given a map of any region (at any location) with a circle drawn on the map (of any radius), there will always be two points on the opposite ends of a diameter that will register the same temperature. T? F? How do we know? (See Graph 6 on previous page.)

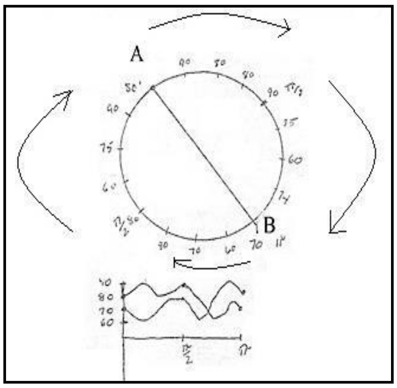
Perhaps we can identify two endpoints, a path to follow, an attribute to measure, and draw a graph to relate these. A diameter drawn on the circle can be used to identify our two endpoints (See Graph 7). Each endpoint has a temperature. This temperature will form the attribute whose change will be measured in this problem. If these temperatures are the same, then our task is over – that is, we will have found two points, the ends of a diameter, with the same temperature. Of course, it is quite likely that the two points we identify will have different temperatures, and like the Hiking Problem, these will be our two distinct starting points.



Graph 7: Weather Map with a Diameter

Consider an instantaneous trip along one of the semi-circumferences of the circle, from A to B, and record the temperatures as you go on Graph 8. Next, take an instantaneous trip along the other semi-circumference of the circle from B to A and record these temperatures as you go on the graph.

Think of A and B as endpoints of a random diameter. If A and B are the same temperature, the problem is solved. If A and B have different temperatures, then consider the graph of temperature change and distance traveled as you move halfway around a circle (moving from point A to point B).



Graph 8: Recording Temperatures while Traversing Semicircles

The solution lies in the fact that temperature is a continuous function (especially when you allow for temperature instrumentation error). That is to say, in a linear sense, if it is 80 degrees HERE and 70 Degrees THERE, then somewhere between HERE and THERE it must be a place where it is 74 degrees. Yes, it COULD also jump above 80 and below 70, but it MUST hit every degree between 80 and 70 in somewhere between. The existence of a diameter with endpoints having the same temperature is “proven” by the intersection of the two temperature graphs. The continuity of temperature ensures that such an intersection will take place at least once. It is not possible to go from X degrees to X + 5 degrees without going through X + 1, X + 2, X + 3, and X + 4 degrees – no matter how quickly it seems the temperature is changing. Combining the Intermediate Value Theorem on continuous functions with one-to-one correspondence of the ruler postulate yields the fact that at least one pair of endpoints to a diameter must have the same temperature.

Conclusion

Problem solving strategies can be useful tools when applying them to attack problems, but they can also be rich tools to explore on their own. The process of reading, analyzing, formulating and verifying solutions takes on a less mechanical air and leads to an experience where the depth of learning is rewarded through transfer.

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PEDAGOGY OF IMAGINATION: EPISTOMOLOGICAL FOUNDATIONS TO DEVELOP MATHEMATICAL THINKING IN PRESCHOOL STUDENTS

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***Abstract**: The recent educational reform in Mexico at preschool and elementary level raises the need to propose pedagogical alternatives for the development of mathematical thinking in children in kindergarten, due to the great deficiencies that are detected in later educational levels. Our proposal is based on the theoretical assumptions of the epistemology of imagination, a post-Piagetian approach that explains imagination as a cognitive process in which practical thinking, imaginative symbolic reasoning, and formal reasoning are involved. In agreement with the human cognitive development proposed by Piaget, we propose as a pedagogical alternative to focus on detonating and fostering imaginative symbolic reasoning in preschool children, instead of mathematical thinking that has not yet been formed or instead of teaching mathematical concepts and procedures such as it was imposed by traditional education.*

Key words: Epistemology of imagination, mathematical thinking, symbolic-imaginative reasoning, pedagogy of imagination, cognitive triad.

INTRODUCTION

In July 2019, once elected the new president in Mexico, it started a transformation of our last educational reform, a political reform but in terms of policies it is similar in its main objectives: a) To give answer to social demand strengthening public education; b) To ensure greater equity in access to quality education; c) To strengthen the capacities of school Management, d) To establish a professional teaching service with rules that respect the labour rights of teachers; e) To promote new opportunities for the professional development of teachers and managers; and f) Lay the foundations for the elements of the Educational System are assessed impartially, objectively and transparently (Secretariat of Public Education, 2017) to fulfil these tasks, the Plan and programs of study are designed for preschool, elementary and high school level condensed all of them in "Key Learning for Integral education containing those normative and pedagogical-curricular aspects useful for teachers of these educational levels. (Secretariat of Public Education, SEP, 2017).

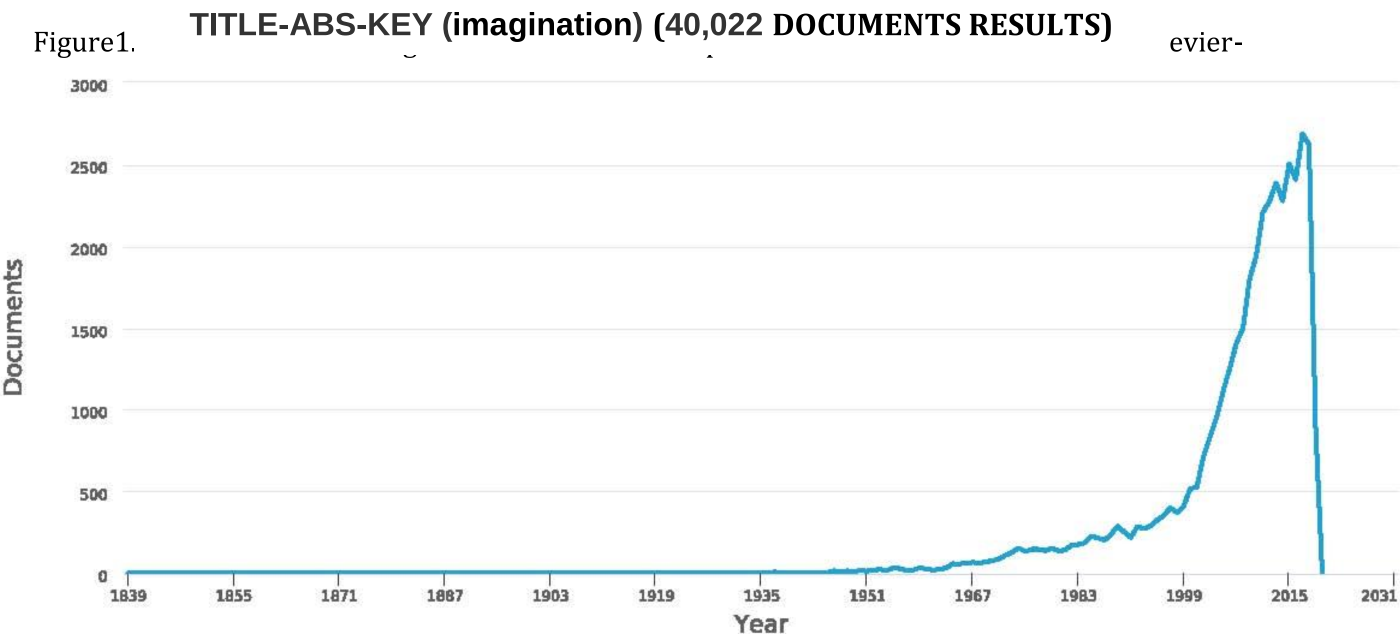
In response to this policy, not politics, in this paper, we present our reflections about SEP vision of mathematical thinking in preschool level. For us, at this level, the students have not mathematical thinking, but a realist and nominalist one, symbolic-imaginative thinking about the external world. Thus, to teach mathematical concepts in preschool children in Mexico, we need to have in mind the child's symbolic-imaginative thinking. The theoretical basis of our proposal is epistemology of imagination, first author theoretical proposal, grounded in Jean Piaget's psychogenesis; a proposal we consider as 21st-century Piagetianism.

THEORETICAL FRAMEWORK

The Secretariat of Public Education (SEP) states that: "Mathematical thinking is a way of reasoning" and its nature often is considered logical, analytical and quantitative. It "also involves the use of unconventional strategies involving novel or creative reasoning "(SEP, 2017:214). In this way, the development of Mathematical Thinking, in preschool children, should encourage imagination for creativity and ingenuity, using daily life natural logic in order to facilitate their adaptation to several circumstances involving the use of numerical, geometric and formal logical reasoning notions for problem-solving.

One of the great challenges for the teacher is to plan "learning experiences" to develop those children's notions. For Rivera & Mendoza (2005), among the most important challenges in preschool education is to define and implement strategies focus in teachers in order to train them and to give them an accompaniment in their professional development that has to be in accordance with the requirements imposed to the education level. Our proposal is the implementation of pedagogy of imagination through what we call the dialogue of imaginations between teacher and their students, as we will see in the next section. In a recent work (Rodríguez-Salazar & Delgadillo-Monroy, 2017), we reported studies imagination since 1839 to 2015, actualized for this work to 2019 with 40,022 documents (figure 1). Specifying search in epistemology of imagination and pedagogy of imagination, fined 302 and 419 documents respectably (figures 2 and 3). Finally, we divided this last by subject area (highlighting the studies in social sciences and art and humanities) and document type, mainly articles and book chapter, books and articles review (figures 4 and 5.

Figure1. ... evier-



Scopus, 2019.

TITLE-ABS-KEY (epistemology AND of AND imagination) (302 DOCUMENTS RESULTS)

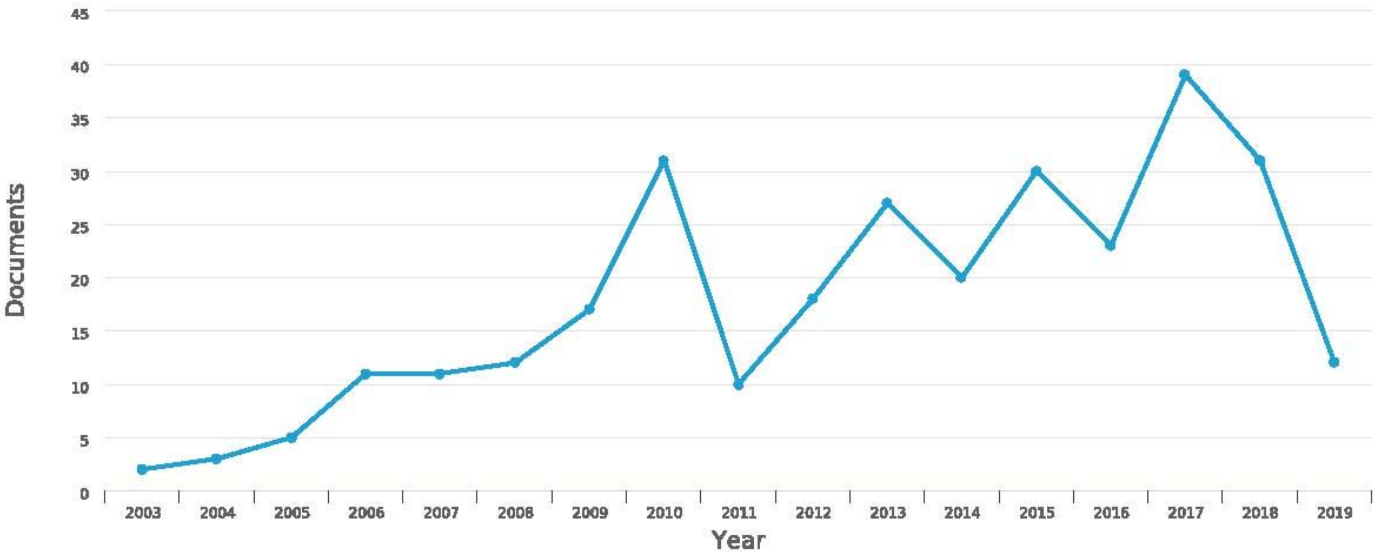


Figure 2. Publications on epistemology of imagination indexed in Scopus form 2003 to 2019: number of publications by year. Source: Elsevier-Scopus, 2019.

TITLE-ABS-KEY (pedagogy AND of AND imagination) (419 DOCUMENTS RESULTS)

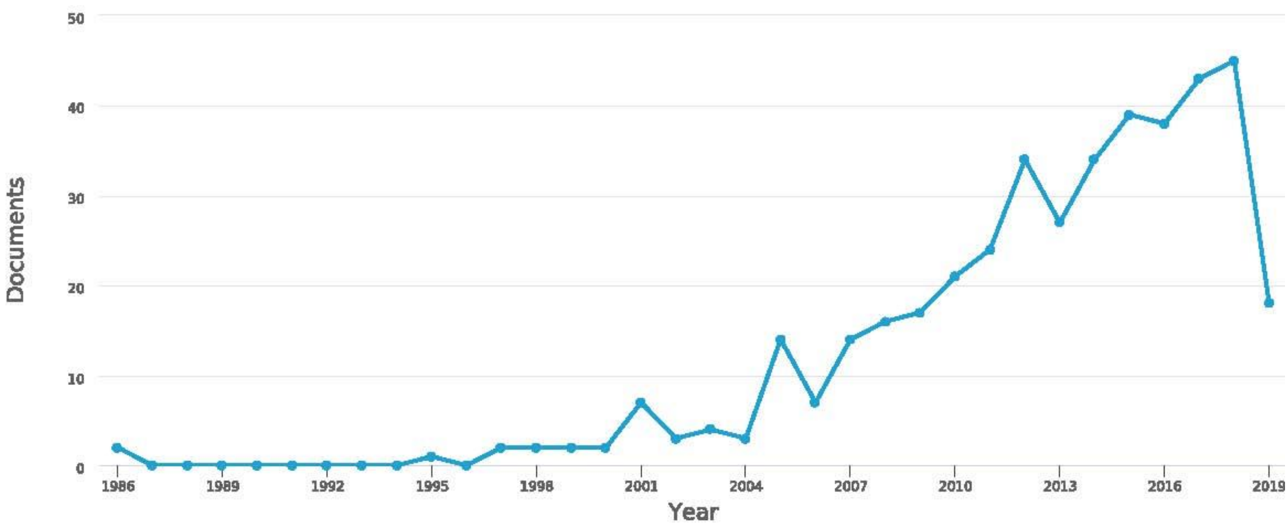


Figure 3. Publications in pedagogy of imagination indexed in Scopus form 1986 to 2019: number of publications by year. Source: Elsevier-Scopus, 2019.

TITLE-ABS-KEY (pedagogy AND of AND imagination) (408 DOCUMENTS RESULTS)

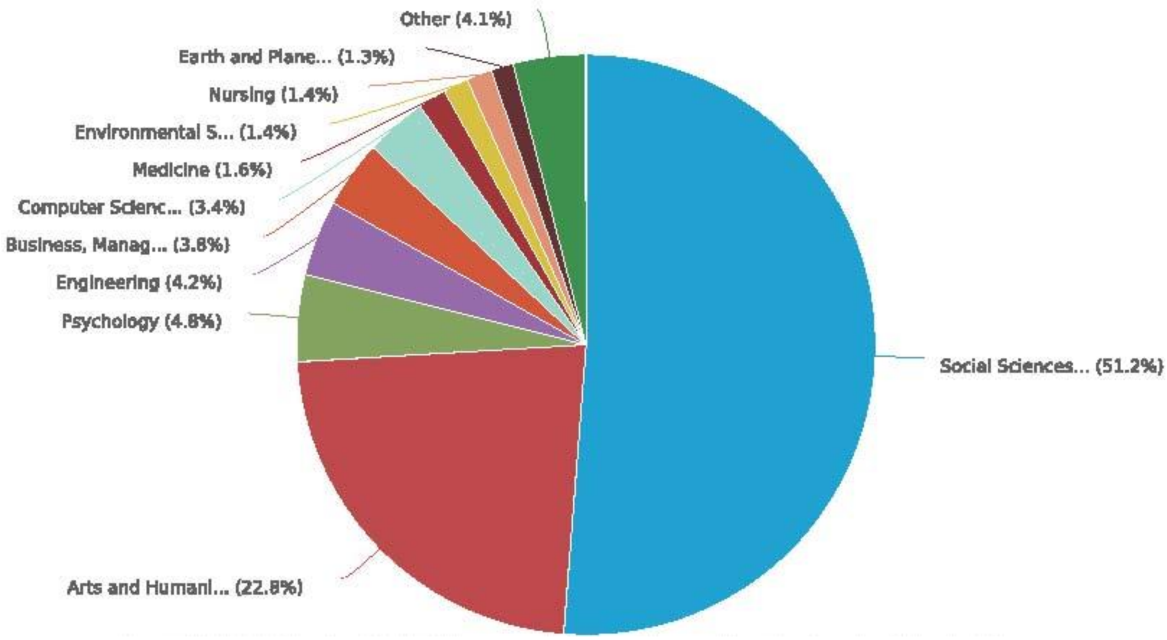


Figure 4. Publications in pedagogy of imagination indexed in Scopus by subject area. Source: Elsevier-Scopus, 2019.

TITLE-ABS-KEY (pedagogy AND of AND imagination) (408 DOCUMENTS RESULTS)

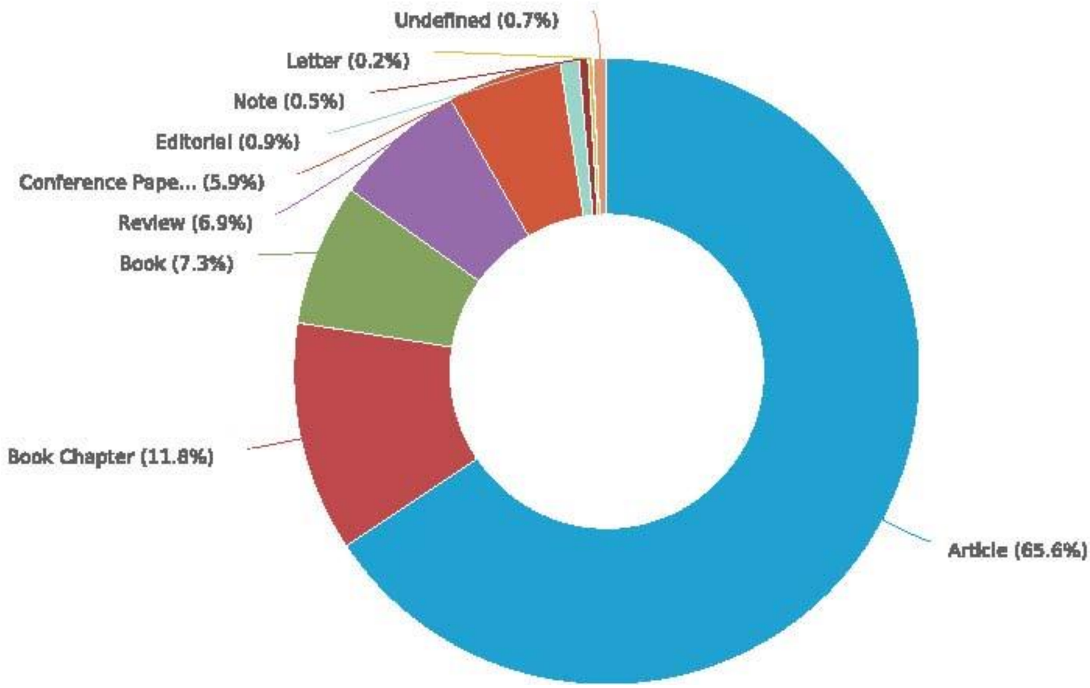


Figure 5. Publications in pedagogy of imagination indexed in Scopus by document type. Source: Elsevier-Scopus, 2019.

In 2016 Strohminger asserts that relevant literature for developing an epistemology of imagination is complex. Valuable work on this field is done by historians, sociologists, philosophers, and cognitive scientists. He also asserts that the epistemology of imagination is only beginning to emerge as a topic in its own right. But not yet it is available a survey about this. Actually, Michael T. Stuart (2019) argues that we should characterize imagination as a cognitive ability, which means cognitive processes closely related to the epistemology of thought experiments. The first and second authors have made several studies about the epistemology of imagination since the beginning of this new century

In the framework, Epistemology of Imagination (own proposal of the first author) acquires two aspects: on the one hand, the development of mathematical thinking by mean of imagination to teach mathematics (teacher) and on the other take advantage of the child's symbolic thinking for the development of logical-mathematical notions (student). The Epistemology of Imagination has been raised as a 21st-century Piagetianism proposal, whose purpose is to develop an epistemology different from the philosophical epistemology. The core proposal of epistemology of imagination is that the scientific adult is integrated by three cognitive structures that we call Cognitive Triad (Rodríguez-Salazar & Rosas-Colín, 2013).

While for Piaget, in adolescence sensory-motor and symbolic thought are subsumed in formal thought as logic-mathematical cognitive structures, for the epistemology of imagination there is not subsumption of these three cognitive structures, but they coexist; there is a coordination between them that allows the subject to organize the world (Rodríguez-Salazar & Rosas-Colín, 2013). As it is shown in figure 3 these structures are: Practical Reasoning (PR), Symbolic-Imaginative Reasoning (SIR) and Formal Reasoning (FR). Then, we propose as a pedagogy of imagination to create a pedagogical dialog between teacher's symbolic-imaginative reasoning and preschool children symbolic-imaginative thinking, transforming realistic and nominalist thought into abstract and conceptual one.

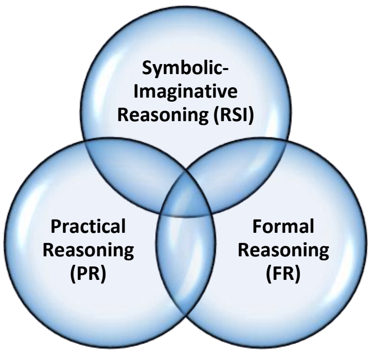


Figure 6. Epistemology of imagination: Cognitive Triad. Source: Rodríguez-Salazar & Rosas-Colín, 2013.

In the framework of the epistemology of imagination, the cognitive triad represents the central axis of our proposal. Thus, in both scientific and teaching practice, imagination constitutes the most important element in the dialogue between teacher's symbolic-imaginative reasoning and children symbolic-imaginative think, designing imaginative configurations throughout those different logical-mathematical structures both in education and in daily life.

DIALOGUE OF IMAGINATION IN PRACTICE

Imaginative configurations elaborated by the children must be used by teachers to create their own imaginary configurations creating a dialogue with students. Thus, teacher could have cognitive "dialogue" by means of reflective abstraction in a process of assimilation and accommodation with the participants. On the one hand the development of logical-mathematical notions and on the other of their mental structures that will gradually allow them to make patterns and organize the world transforming imagination in formal actions.

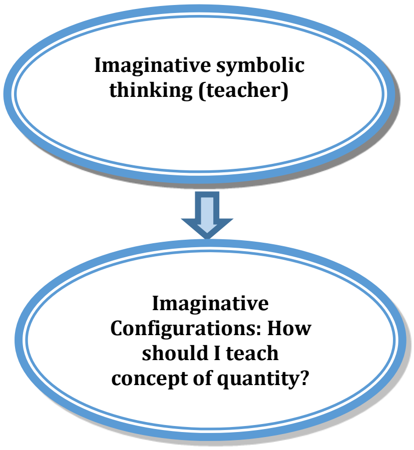


Figure 7. Learning experiences based on imaginative configurations. Sours: Own elaboration

We consider that the Epistemology of Imagination is the base of a 21st century Piagetianism constructivist pedagogy, while allowing the development of mental structures of the teacher and the saying, and also favors the assimilation and accomodation of new knowledges. Moreover, being the internal epistemology of the field of mathematical thinking formation, it may well be an epistemology derived from the other fields of academic formation that integrate the Plan and programs of study of basic education, because the symbolic imaginative thinking is by its very nature transversal to any educational practice.

Indicated out the theoretical basis on which our proposal for the Creative Teaching (EC) of mathematics rests, we will go on to develop general aspects on the method for the elaboration of a program of reinforcement of the symbolic thought Imaginative for preschool teachers.

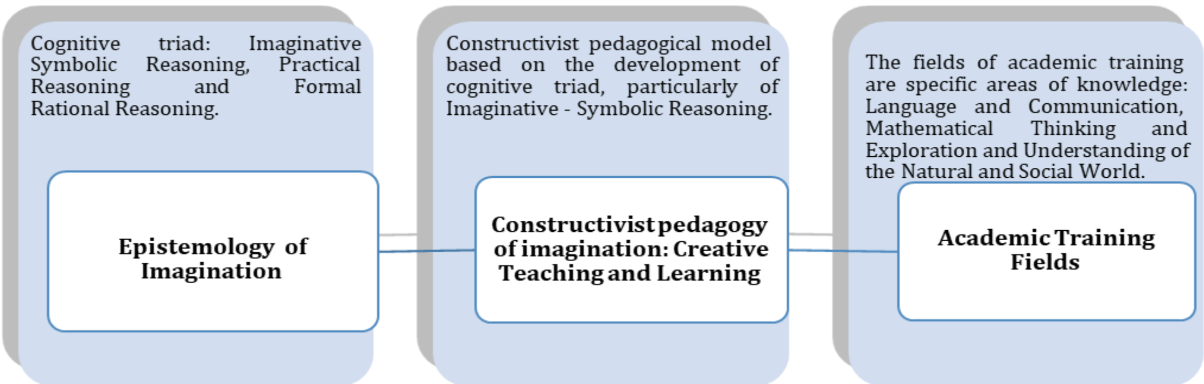


Figure 8: Constructivist pedagogy of imagination based in epistemology of imagination. Own

General Objective: Teachers will develop didactic resources to favor the imaginative symbolic reasoning of preschool realistic and nominalist children think toward the development of mathematical thinking.

According to the psychogenic theory of Jean Piaget, at this level, children's thinking is on things (realism) and on what it is named by words (nominalism). In other words, the children's thinking is in the contribution of the adult to the child's belief. Then it is necessary that children's thinking detached from things, through abstraction process: fundamentally to move from thinking in general to mathematical thinking.

The example that we are going to use for the concept of quantity is the number four, with four different objects, modifying its organization by means of addition and subtraction. For example, four apples together; two apples adding one apple; three apples adding, going after, to six apples less two, five apples less one, etc.

Figure 5**:** Constructivist pedagogy of imagination based in epistemology of imagination. Own

<table>
<tr><th colspan="2">Target population: 20 teachers of kindergarten</th></tr>
<tr><th colspan="2">Approximate execution time: 1 session per week for one month</th></tr>
<tr><td>The teachers will take a place in the work room, in such a way that they have enough space to work in harmony. Each of them will be provided with two white sheets, a pencil, a rubber, pencil sharpener and colors.
Later, they will be given the following indication: "In the leaves they have drawn an apple" There will be no time restriction, nor additional indications regarding the size, color or position of the sheet.
When all the teachers have finished the drawing they will be asked to answer the following questions. What did they draw? Where is the name of what they have just drawn? The answers must be written on the same page on which they made the apple.
On the second sheet you will be asked to draw four apples. There will be no further indications regarding the size or color of the drawing, or any other type. The teachers will make the requested drawing and at the end they will answer the following questions: How many apples have they just drawn? Where is the number that represents that number of apples? The answers will be placed on the same page.
They will be asked to represent on the same page the number of apples they made in the previous year, but now in relationships that involve adding; for example: 3 + 1, 2 + 2, etc. Finished the exercise, they will represent the same amount but now in relationships that involve removing; for example: 8-4, 10-6, etc. Each teacher must place at least three examples of each of the indicated relationships</td><td rowspan="3">Expected results: That teachers can reflect on the influence of their imaginary configurations in the teaching process, and that this in turn allows them to understand the importance of symbolic-imaginative reasoning in the design of learning situations that allow them to "dialogue" their configurations imaginary with those of the students.</td></tr>
<tr><td>Instruments used: Educational planning of the Academic training field of mathematical thought. The imaginative configurations of children; regarding the conventional and unconventional use of mathematical knowledge in several learning experiences, will also be taken into consideration.</td></tr>
<tr><td>Challenges encountered: The professionalization and updating of preschool teachers has resulted in a kind of "bombardment" of information that is wasteful for them. One of great challenges will be to generate interest so that they have openness in the realm of imagination and creativity, far removed from the institutional bureaucracy</td></tr>
</table>

FINAL COMMENTS

Today, attention is directed to the strengthening of the teaching practice, and especially to the creation of creative and attractive learning experiences for children. It is thus proposed to bet on an innovative and more dynamic teaching. Therefore, our proposal, invite to the development of imagination through the dialog between teacher and students offering a perspective since epistemology of imagination by pointing out the coexistence of three types of reasoning. The connection of both is symbolic-imaginative rezoning. It is therefore an approach in which both educational actors have an active role and whose main objective is for the student to develop his mathematical thinking to respond to the circumstances of daily life.

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TASKS ON VISUAL PATTERNS AS THE FIRST STAGE OF INTRODUCING ALGEBRA CONCEPTS

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Abstract. *The presented paper discusses specific questions of leading mathematical circle for lower grade students at the Correspondence School of Mathematics at the University of Latvia. The paper covers the aspects of solving tasks on visual patterns, the importance of which is recognized by many researchers. Such tasks should be one of the first foundational stages in acquiring the methodology for solving Olympic problems. The paper provides some sample problems and discusses students' solutions. Excerpts from teacher's notes demonstrate active student participation in class where they learn from each other's ideas.*

Key words: Algebraic reasoning, mathematical circles, problem solving, tasks on visual patterns.

INTRODUCTION

Today mathematical contests of various kinds are popular worldwide: hundreds of competitions are organized on a national, regional and international level every year (Kenderov, 2009). New forms of contests are available thanks to the evolution of computer technologies. Let us mention some of them. Online mathematical competition is offered by Global Math Challenge (https://www.global-math.com/top) for schools or for any individual person; competitions for students of any grade are organized by Mathematical Kangaroo (https://www.mathkang.org); the Centre for Education in Mathematics and Computing at the University of Waterloo (https://www.cemc.uwaterloo.ca/) organizes a local Canadian contest that welcomes participants from foreign countries. Many events have become well known through personal attendance of contestants. The International Mathematical Olympiad for high school students (https://www.imo-official.org/) ranks most prestigious among them.

Any student who is willing to succeed in the competitions has to make significant effort, as is done in music, art, sport, and other activities. One way to enhance performance is participation in a mathematical circle. Mathematical circles have particular goals and different forms. Some of these extracurricular activities are focused on training students for mathematical competitions, on giving students the opportunity to develop their problem-solving skills and habits of the mathematical mind ("What is a Math Circle?" n.d.; McCullough & Davis, 2013), or on getting school children interested in mathematics and encouraging creative thinking. Most of the circles involve students starting from secondary school grades. The number of events targeted toward lower grade students is significantly less (Mast, 2015; McCullough & Davis, 2013). However, the situation has been changing in the recent years – circles and clubs for young children are being organized all over the world. The mathematical clubs in South Africa involve young students from second to fifth grade and are intended to develop children's proficiency and to promote their interest in mathematics (Stott, 2016). The project PriMa at the University of Hamburg started in 1999 (https://bildungsserver.hamburg.de/00-np-prima/) and targets mathematically talented 3rd and 4th grade students. Stanford University offers an elementary mathematics circle to any mathematically motivated students from 1st to 4th grade. Circle leaders and researchers create extensive support materials for the teachers and parents who are willing to present exciting ideas to young children. Anyone interested in this field can find very well planned lessons with exciting videos (https://www.youcubed.org/weeks/week-1-grades-3-4/). Series of books exploring the methods of teaching, learning and problem-solving are available. They include sets of problems that are elaborated after their practical application in the classroom - four volumes of the book "Schülerzirkel Mathematik Grundschule" were published as a compendium of PriMa project from 1999 – 2016 and can be downloaded free. The importance of the development of mathematical ideas for preschool children is recognized as well. Zvonkin (2011) shares his experience of leading a mathematical home-circle for children that were about 4 – 5 years old. He wrote a journal about the problem-solving where he commented on children's reactions with psychological explanations.

THE IMPORTANCE OF INVESTIGATION OF VISUAL PATTERNS

Many researchers and educators point out the benefits of visual interpretations of mathematical statements in the teaching-learning process of mathematics. Methods to describe particular mathematical results by visual pictures have been known since ancient times (Fried, 2010). Nelsen (1993) explains the pictures "Proofs Without the Words" as a tool that helps the observer see why a given statement may be true, and shows how this statement can be proved. Arcavi (2003) writes about different forms, uses and roles of visualization in mathematics education. Demonstrating the problem on matchsticks counting as one of examples, he discusses different combinatorial ways how to deconstruct a given picture. He concludes that visualization of a problem can be considered an analytical process itself, which leads to a general formal solution. Boonen et al. (2106) point to students' difficulties in solving non-routine word problems. They note how important it is to construct an accurate visual-schematic representation to describe a given word problem. Gierdien (2007) reports on the usefulness of visual interpretations of series and sequences at the secondary school classes. He notes that interpretation of numbers as geometric figures and constructions thereof includes the ideas of the analytic proofs of theorems and of algebraic formulas. By the investigation of pictures, students can better understand the analytical processes.

Barbosa et al. (2012) find that pattern activities and visualization can be significant aspects of the learning process. Pattern exploration tasks are useful to develop students' abilities to organize data, to make conjectures, and to generalize. Boaler et al. (2016) report on new brain research that reveals the importance of visual thinking. Neuroimaging shows complicated communication between widely distributed networks of memory, control, detection, and visual processing during mathematical thinking. Tavares and Cabrita (2014) focus on the importance of visual exploration of patterns in the development of students' creativity. They point to different approaches to explore given tasks that can be constructive generalization, where the pattern is composed in several subsets and expressed by the additive formula, and deconstructive generalization exposed by numerical expression with subtraction.

TASKS ON VISUAL PATTERNS IN THE MATHEMATICAL CIRCLE

The Correspondence School of Mathematics (CSM) at the University of Latvia has organized a mathematical circle for lower grade students. The aim of the circle is to increase students' interest in mathematics, demonstrate the extensive content of Olympic problem sets, and teach the students the first steps of problem solving. Once a week, 4th and 5th grade students can take a part in the "Beginners" class, while 6th - 8th graders can participate in the "Continuators" class. The content of classes is varied, including problems from different fields of mathematics – number theory, arithmetic, logic, combinatorial geometry, and others. Here the main focus lies on the formulation of tasks. The description should be attractive and understandable to engage children in the solving process.

Solution of different Olympic problems is based on the skills of grouping given elements by common properties and making numerical evaluations, where application of algebraic calculations is an important tool. The description of givens in algebraic terms and expressions is crucial to be able to describe solutions in accurate mathematical language.

Lower grade students are not familiar with the algebraic notions yet. However, tasks on visual patterns can be applicable to develop counting methods. Students can learn how to translate them into numerical expressions. As Vale and Cabrita (2008) note, the investigation of visual patterns can form the foundation of algebraic reasoning and looking for pattern can be a useful strategy in problem solving.

At the beginning students should learn to investigate given figures and visual patterns and to formulate questions. Thus, the mathematical circles of CSM include such tasks in the classes by setting the following goals:

- To find common properties of substructures in a given figure
- To discover different methods of counting
- To find the invariant properties of elements in the sequence of given figures
- To create an algorithm of changes
- To try finding an original solution of a problem.

EXAMPLES OF SOLVING THE TASKS ON VISUAL PATTERNS

Example 1

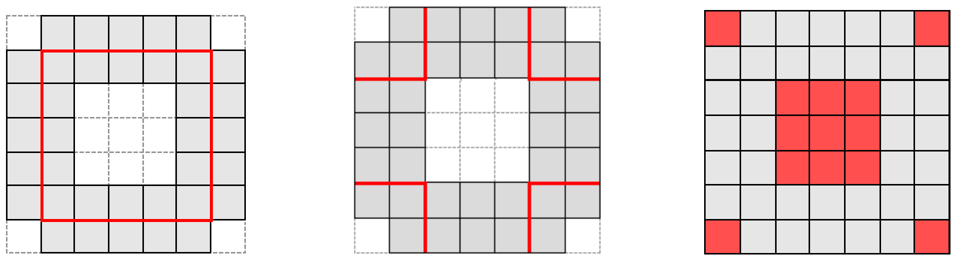
Every student in the class receives a list of problems to be solved one by one. First, the current problem is solved individually or in pairs; then the solution is discussed by the whole class. If students "get stuck" on a solution, the teacher turns their attention to the givens described in the problem and asks questions to facilitate students' activity for investigation of regularities of the givens.

Problem 1 ("Beginners"; class 22; 24 participants). The first three figures of the sequence are given in Figure 1. How many unit squares can you count in the 10th figure?

Figure 1: Figures constructed from unit squares

Some of the students construct the sequence of the following figures and simply count the squares with numbers 20; 28; 39; 44 … This caused the following questions: what are the regularities of this number sequence? How to calculate the next number without drawing the figure?

Other students used deconstruction methods separating the figures into blocks (see Figure 2).



a) b) c)

Figure 2: Separation of figures into blocks

In Case a) four side bars and the frame are displayed. For the next figure the bar is longer by one square and so is the frame. The 10th figure has bars 12 squares long and the frame is from the square with a side length of 12 as well. Students calculated the number of squares in the bars $4 \cdot 12 = 48$ and counted the squares in the frame. For Case b) two types of blocks were detected: rectangles with size $2 \cdot n$ and L-shaped figures in the corners. In Case c) a completely different method was used: the figure was complemented until the square was full. For the 10th figure the side length of the complemented square is 14 unit squares. The number of unit squares in the figure is $14 \cdot 14 - 10 \cdot 10 - 4 = 92$.

The demonstration of these outcomes to the whole class allowed students to experience original solution methods. As stated by Tavares and Cabrita (2014), different ways of construction and deconstruction of figures foster students' creativity.

Example 2

The following lesson is about figures constructed on a triangular grid. Introductory tasks are about different ways of counting the elements in the figures. For the "Beginners" class, the main goal is to detect different counting algorithms and to describe them by numerical expressions. The students in the "Continuators" class have to detect general algebraic regularities of a sequence of figures and create an algebraic description of the process. Below there are two short excerpts from the teacher's field notes and an example of algebraic reasoning from the higher grade students.

Problem 2 ("Beginners", class 11, 29 participants). Count all edges within the hexagon with the side length 2 in Figure 3! An edge is a line segment that connects two nearest points.

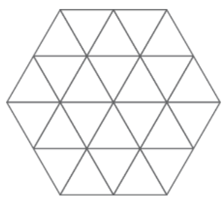


Figure3: Hexagon with the side length 2

Protocol 11, "Beginners", Excerpt 1 (Children's names are fictional)

Children start to work by counting edges one by one.

Anna:	I counted 40!
Emil:	No, there are 42 edges!
Teacher:	How can you be sure that the number is correct?
Anna:	Hm … I can colour each edge! … Yes, I was wrong.
Teacher:	Well done! But what if you try to count the edges in the hexagon with side length 100?
Anna:	I will colour the edges one by one.
Paul:	Can you imagine how big such an example is? It does not even fit to the given sheet of paper!

Comment. The class discovered two methods of counting: by examination of all parallel edges and by examination of borders of concentric hexagons and edges between them. In the first method, the set of parallel edges contains 2 + 3 + 4 + 3 + 2 = 14 units (see Figure 4.A). As the edges are oriented in three directions, there are 3 sets of mutually parallel edges, 42 in total.

Another approach found by students counts the edges on the borders of hexagons and the edges between the borders (see Figure 4.B).

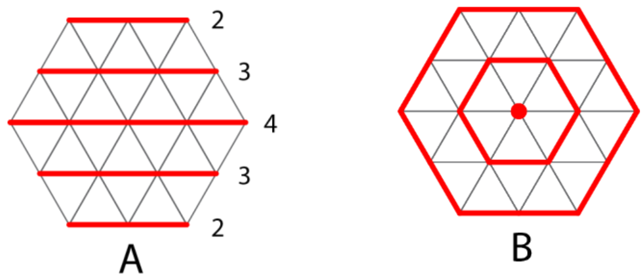


Figure 4: A - Set of parallel edges; B - Edges on concentric borders

Problem 3 ("Beginners", class 11). Count all unit triangles inside the hexagon with the side length 5!

Protocol 11, "Beginners", Excerpt 2

Elsa:	There are two types of triangles – with the vertex up and with the vertex down. I will count the triangles with the vertex up first.
Anna:	I will colour these triangles! … They lie on the edges, so I count …. 6 + 7 + 8 + 9 + 10 + 9…. 75!
Emil:	The number of blue triangles in the upper half is the same as the number of white triangles in the lower part!
Michael:	I found a simpler way of counting! I separated the hexagon into six equal parts, one part contains 25 triangles – it is easy to count. They are 150!

Alex:	Teacher, teacher! Now I see the way to count the edges! Colouring like a chessboard shows that the big triangle contains 15 small black triangles that use all the edges. There are 45 edges, and the hexagon has six parts, and all the edges are 45 times 6 … 270!
Teacher:	Is this the same number that we counted before?
Michael:	Big triangles have common edges, you counted some edges twice!

Comment. Michael's insight about separation of the hexagon led to an additional discussion. The teacher introduced the class to quadratic numbers and their triangular interpretation.

Problem 4 ("Continuators", class 11, 17 participants). Find the formula for calculation of the number of edges of a hexagon with the side length n!

Students found the formula by applying two algorithms of counting. *The first method* was to investigate the sets of parallel edges in the hexagon. Students noted that one set can be divided into two equivalent parts on both sides of the diagonal. Understanding that the hexagon can be separated into 6 equilateral triangles, it was easy to see that the diagonal of the hexagon is twice as long as its side. By counting the edges on one side of the hexagon, the following formula can be made:

$$(n + n + 1 + n + 2 + \dots + (n + n - 1)) \cdot 2 + 2n =$$
$$= (n^2 + \frac{1 + n - 1}{2} \cdot (n - 1)) \cdot 2 + 2n = 3n^2 + n$$

The total number of edges in all three sets:

$$(3n^2 + n) \cdot 3 = 9n^2 + 3n$$

The second method was based on the properties of triangles. The sixth part of the hexagon includes n^2 unit triangles. This part can be coloured like a chess board (see Figure 5). Let us suppose that the first line of black triangles lies on n edges. Then the number of black triangles is:

$$n + n - 1 + n - 2 + \dots + 1 = \frac{n(n + 1)}{2}$$

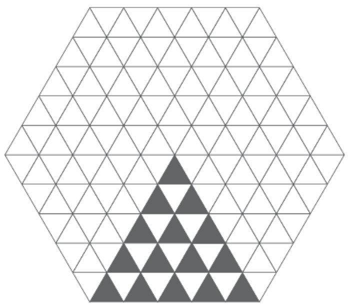


Figure 5: Colouring of triangles

The total number of edges of all triangles in all six parts of the hexagon is:

$$\frac{n(n + 1)}{2} \cdot 3 \cdot 6 = 9n^2 + 9n$$

In that way the edges on diagonals are counted twice. Therefore, the number of edges in the hexagon is:

$$9n^2 + 9n - 3 \cdot 2n = 9n^2 + 3n$$

The proof of the detected formulas is quite a difficult process and is a topic for high school students. Therefore it is not included in the classes of the CMS mathematical circle.

DISCUSSION ABOUT THE WORK IN CLASS

These small excerpts from the introductory part of the lesson show how the students' comprehension about problem-solving changed. The students started to solve the task by a commonly recognized method – the full enumeration method. Then their "eyes opened" and they discovered different properties and interconnections of the elements step by step. Discovered methods reveal the algorithms applicable for calculations of bigger examples. Discussions between students created an open and friendly atmosphere in the class. The sharing of opinions allowed the students to experience different viewpoints on the given tasks. They were able to learn that problems can be solved in various ways that foster students' fluency and flexibility in problem-solving. Excerpt 2 presented above demonstrates Michael's delving into the problem and the original insight that caused Alex to backtrack to the previous problem. Students' collaboration helps to elaborate new ideas and to improve the solutions of tasks.

Some students in the "Continuators" class participated at the CSM circle in previous years and had obtained more experience at participating in mathematical Olympiads. Students applied the well-known formulas of the sum of number sequence to simplify the counting. They implemented the principle "to look at the problem from distinct points of view". In the subsequent part of the class, challenging combinatorial problems on figures on a triangulated grid were presented. Different ways of calculation here were really helpful to check the hypothesis stated in class.

CONCLUSIONS

Investigation of visual patterns and detection of regularities introduce students to the main principles of algebraic reasoning. Discovering different counting methods develops students' analytic and algorithmic thinking, and they can better understand the essence of algebraic formulas. Understanding of relations in visual patterns stimulates students to draw visual schematic representations of various word problems themselves. As noted by Boaler et al. (2016) students are excited to see the growing patterns, they engage in the problem and learn deeply about functional growth. Researchers point to the significance of teaching students to use cognitive strategies for solving non-routine word problems where accurate visualization of givens is one of the core methods. Boonen et al. (2016) discuss the instructions on teaching students to visualize the structure of a problem to reveal relations between the relevant elements.

Tasks on visual patterns for lower grade students are useful both for classes of mathematics circles and classes of compulsory curriculum, too. The solving of such tasks strengthens students' conceptual understanding, forms strategic competencies, and fosters creativity. Algebraic reasoning reveals student's proficiency in problem-solving. These skills are valued at mathematical contests and Olympiads.

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CREATIVITY, TECHNOLOGY AND "OUT SCHOOL" INTERESTING MATHEMATICS WITH TECHNOLOGY DURING OUT SCHOOL

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Abstract. *Mathematics should be a creative endeavor and enjoyable pastime throughout one's life, not just during school and careers. This paper describes the basis for this and gives examples of interesting problems that can be investigated by anyone at home using readily available technology resources. Examples from a mathematics workshop for adults who are out of school and looking for interesting leisure pursuits are also given. The term "out school" is created to describe these activities.*

Key words: creativity, mathematics, lifelong learning, technology

"OUT SCHOOL" AND ENJOYMENT OF MATHEMATICS

The research on mathematics education is biased to only include school education. In the Japanese case, most people stop learning mathematics when they graduate from school. It is a very sad problem for all. Why are they stopping learning mathematics? There are some reasons that people stop learning mathematics. One is that many students dislike mathematics at school. This fact has been verified by PISA (Programmed for International Student Assessment). If students like mathematics, they will want to learn more mathematics after school. At school, they may learn mathematics with logical thinking, and often are given mathematical knowledge by teachers. They lose their sense that mathematics is a pleasant subject. We define this system using the term "In School". For "In School" education, we have two parts, one is the teacher and the other is the student. Teachers have knowledge of mathematics and are teaching it to students. Students often just receive what is given as mathematical theorems. It is evident that such an educational system may make students hate mathematics.

To define "Out School", many people will say "Out School" is Lifelong Learning. In Lifelong Learning, if someone goes to school for acquiring the knowledge to do a new job, I do not say this learning is "Out School". Going to school, he studies with teachers. "Out School" learning may be self-taught with technology. And in "Out School" learning, we use inductive thinking by doing mathematical experiments. We want to do mathematics with enjoyment. So, the aim of learning in "Out School" is enjoyment. What is mathematics for some people? It is important to ask them. I think doing mathematics is the same as thinking. This thinking is inductive, not deductive. Additionally, "Out School" mathematics does not need assessment by teachers. For inductive thinking is near creativity. I show this move to the next level of mathematics in Fig.1.

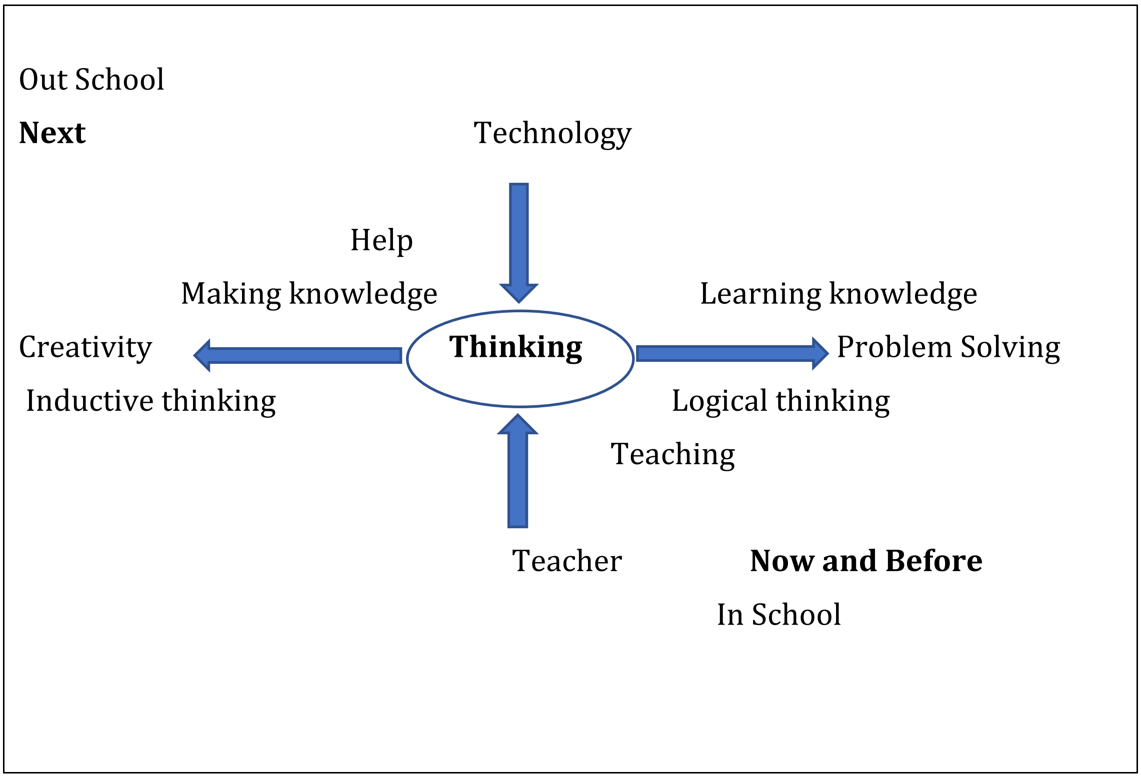
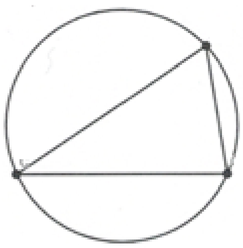


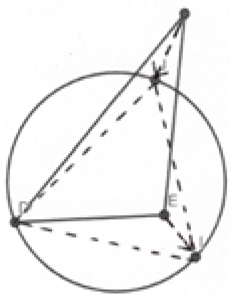
Fig.1: “Out School” and “In School”

GEOMETRY EXAMPLE OF “OUT SCHOOL” LEARNING MATHEMATICS WITH TECHNOLOGY

The following theorem is well-known. So, this theorem is the beginning of other problems. Using a circumferential angle and moving points, we want to find out new interesting facts. Anyone can use the free online math app GeoGebra or similar online math tools.

Theorem: The sum of the interior angles of a triangle is 180 °





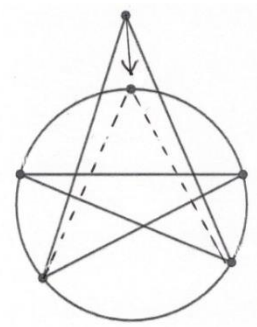
All triangle vertices on circumference

(Any three points are on circumference)

Using circumferential angles, we get 180°

no point on circumference

moving this point to circumference



All triangles on circumference

(Any three points are on circumference)

Using circumferential angles, we get 180 °

no point on circumference

moving this point to circumference

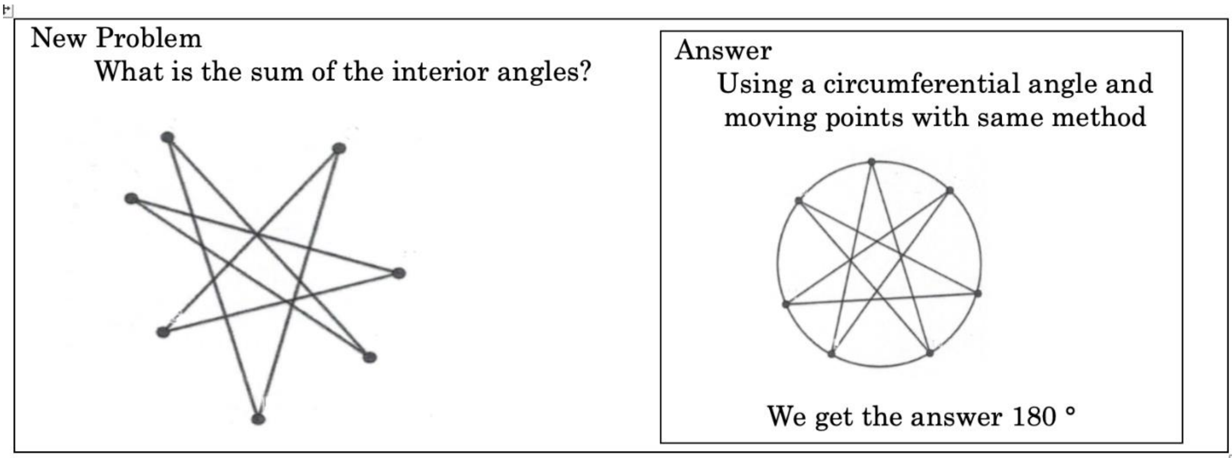
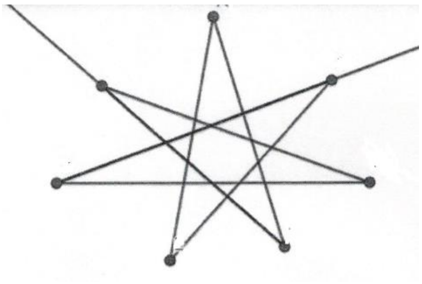
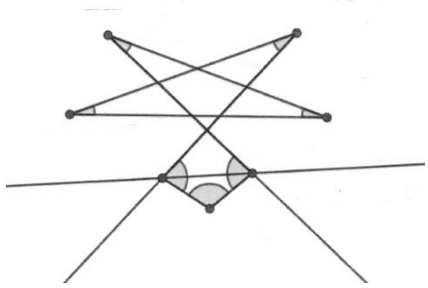


Fig.2 New method of solving problems

We are interested in solving this problem, and we can get a new problem. In this case, we use only seven points, but if we draw the n-star polygon with a stroke of writing, then the sum of the interior angles is still 180 °. Doing mathematics may be finished by getting an answer to this problem or we might continue exploring. We are interested in thinking about mathematical problems with technology. When we are solving this problem, there was not any teaching or advice from the teacher. This activity is “Out School”.

We are interested in researching of our newly-found problem. We can’t build up a good solution if we are afraid of the possibility of always failing, but there must be a possibility of including errors or insufficient reasoning when we are self-taught. If our mathematical thinking includes fear of mistakes in this problem and answer, we can’t enjoy this “Out School” activity. One thought may lead to failure when learning mathematics, but negativity cannot stop the exploration. In searching for a solution, it may be fun and painful to want to solve the problem when we learn that we do not have the security that a point can move anywhere. We have new problems and want to think about them. We try to explain this method, and using technology can help. As a result of mathematical experiments, we can easily know the range that points can move. We can see the value of knowing the sum of the interior angles is 180 °. And with some experimentation, we find other interesting positions of points. We can see they do not change the value of sum of the interior angles. What is meaning of this value? We have more problems to think about.





This region is the sum 180°   This region is not changing value

This example is an easy problem for mathematicians, but thinking and creating is an interesting "Out School" activity.

ALGEBRA EXAMPLE OF "OUT SCHOOL" MATHEMATICS USING TECHNOLOGY

Using technology, we can enlarge the world of mathematics. Seeing mathematics is interesting. The following example is from algebra. By looking at mathematics with technology, we feel like we can make a hypothesis and conjectures.

Conjecture 1: $(1/9)^{n}$ recurring decimal has a period of $9^{(n-1)}$

1/9 and 1/9*1/9 change to repair it in a recurring decimal

1/9=0.111・・・・・・・・・・period 1

1/9*1/9=0.012345679012・・・period 9

1/9×1/9×1/9・・・・・・・・period 81 = 9^{2}

1/9×1/9×1/9×1/9・・・・・ period 243 = 9^{3}

We want to find a solution for the period of (1/9) ^n, but using technology such as a stand-alone calculator or a calculator on the phone or computer, we can't see more. If we get a larger number n, there is no readily-available technology that will show the computation. The world of mathematics is not concrete, it must be abstracted. If the conjecture is proved, we can create a new theorem. But it is difficult to prove this conjecture. We want to find out the role by seeing concrete figures. It is interesting to find out the role. If there is no proof of this conjecture, a mathematician might be sad for not making mathematics. But as an "Out School" activity, we can enjoy the exploration every time, and dream about proving it.

We can't prove the mathematical theorem, so we change the number 9 to 7.

1/7=0.142857-・・・・・・・・・period 6=6×7^{0}

1/7*1/7=0.0204081632653061224489795918367346938775510・・-period 42=6×7

1/7×1/7×1/7・・・・・・・・period 294 = 6×7^{2}

1/7×1/7×1/7×1/7・・・・・・ period 2058 = 6×7^{3}

So, we can make the same conjecture,

Conjecture 2: $(1/7)^{n}$ recurring decimal is period $6\times7^{(n-1)}$

And we can change Conjecture 1 to adding $1\times9^{(n-1)}$.

Conjecture 1': $(1/9)^{n}$ recurring decimal is period $1\times9^{(n-1)}$

We change the conjecture from $(1/9)^{n}$ to period $1\times9^{(n-1)}$. The number is same, but the role is same for the conjecture 1' and 2. We finish this activity, no proof and no theorem, but fun to explore.

But one more exploration, seeing the number,

0.0204081632653061224489795918367346938775510

we can explore the role of rearranging the number.

02	04	08	16	32	65	30	61	22	44	・・・・
2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	・・・・
					64	28	56	12	24	
Adding number					128	256	512	1024	2048	
Same					65	30	61	22	44	

This arrangement is the number 1/7*1/7=1/49=2/(100-2). We want to calculate 3/(100-3). And 4/(100-4).

3/(100-3). =0.030927・・・

4/(100-4). =0.041666・・・

Seeing these numbers, we can make another conjecture.

Conjecture 3. n/(100-n)=0.(0n)$(0n^2)(0n^3)$・・・ all numbers are two-digit numbers

It is easy to see that this hypothesis is erroneous. However, it is possible to find an interesting rule hiding in numbers, and the pleasure of following that rule is considered an important mathematical activity in "Out School".

CONNECTING LEARNING MATHEMATICS "IN SCHOOL" TO "OUT SCHOOL"

In Japan, many people finish learning mathematics when they graduate from school, and they say that they disliked learning mathematics in school. They are learning mathematics only "In School". In Japan, there is nothing "Out School". I want to train people to sustain lifelong mathematical learning. Since 2010, I have led training sessions for adults. This year marks the Tenth International Conference on Lifelong Learning Thai and Japan. The aim of this is not teaching mathematics, it is thinking it. This conference is designed to connect "In School" mathematics learning to "Out School".

In this session, we begin with telling ourselves to look for mathematical nomenclature contained therein. It can be any small thing while considering mathematics, where the goal is neither learning nor learning new mathematics. Using mathematics is very easy to understand. We want to create our small mathematics. We do not need a big imagination, but we want to experience the pleasure of thinking mathematically by getting small creations. "Out School" mathematics is enjoyable for us. Enjoyment is an important word for "Out School". We want to enjoy learning mathematics for a lifetime. But most adults do not know what "Out School" learning is. So, I am training adults to learn mathematics without teachers, and we want to create new small mathematics. The following is one example of a training session.

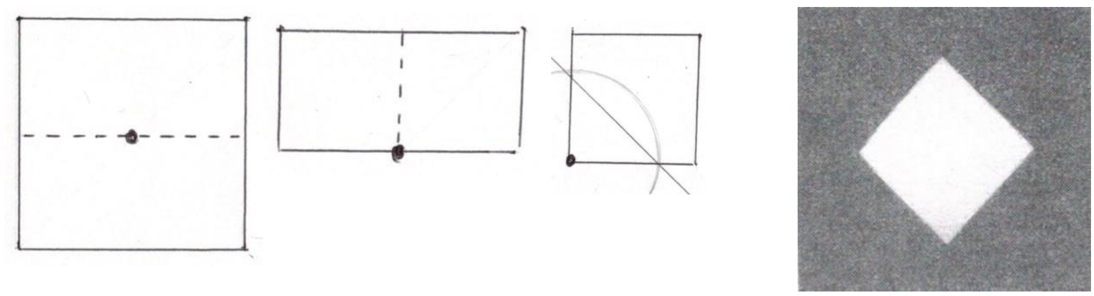
Title: Making Polygons from Folding Origamis and Using One Cut

Target: Adult people 50- to 70-years-old

Time: Two hours

Gathering: 15 persons

First Step: Fold and cut



Cut with one cut on line

Origami folding at dot line Getting square

Second Step: Discuss after seeing this square

Understanding Symmetry

Vertices on the dot line…it is very important to think about making polygons

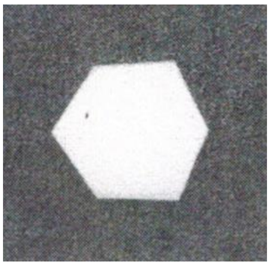
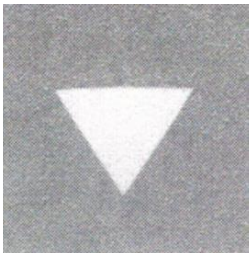
The edges and place

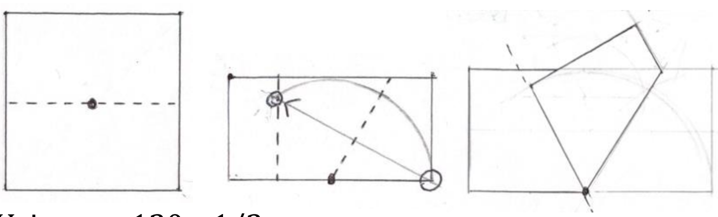
The angles and place

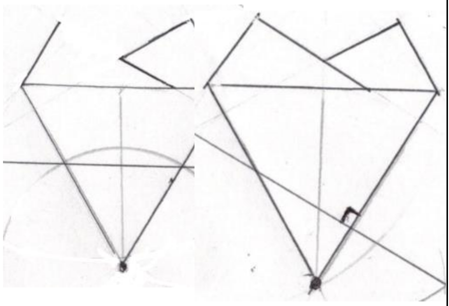
The relation of vertex numbers to folding times - 2 fold times then 4 vertices

We make the conjecture 2×fold times=number of vertexes…This in not collected.

Third Step: Create a regular hexagon and a regular triangle

Regular triangle  hexagon  Regular





Using cos 120=-1/2

Regular hexagon Regular triangle

What we are thinking is not to learn new things but to experience fun by thinking. In this session, we can get pleasure from thinking ourselves. In this workshop, I think that we were able to achieve the connection from "In School" to "Out School" that we targeted.

The showing of the training session

THE AIM OF MATHEMATICS: LEARNING "IN SCHOOL" AND ENJOYING "OUT SCHOOL"

What is the meaning of learning mathematics in school education, "In School"? And what is the necessity of learning sustainable mathematics through lifelong learning, "Out School"? Mathematics is an important subject in school education, and all students are taught it. It may not change in the near future. The problem imposed on us now is the importance of continuing to see the beauty of mathematics for the rest of our lives and enjoying life by learning it. We need small creativity for all, so they enjoy their lifetime. Lifelong learning is continuous learning, but we don't always know how to do it. We need to revise our attitude that teachers will always be needed for learning. In lifelong learning, there is no teacher. We are the students and the teachers. We lead our lifelong learning in "Out School". It is obvious that there are many differences between school learning and lifelong learning, but it may be difficult to distinguish lifelong learning from "Out School" learning. There are differences in thinking about lifelong learning that are considered in many countries. I want to have a pleasant life by learning mathematics even after school education is over. Learning mathematics should be enjoyable for all and for all times. For this purpose, small creativity is very important, and it can be greatly enhanced through the use of technology.

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Notice

Using Technology is GeoGebra and TI-89.

WORKSHOP

MATHEMATIZING CREATIVE STEM PBL ACTIVITIES

Mary M. Capraro, Robert M. Capraro Katherine N. Vela, Cassidy Caldwell, Danielle Bevan, Yujin Lee
Aggie STEM & Texas A&M University

Abstract. *Prior research has indicated that the use of science, technology, engineering, and mathematics (STEM) project-based learning (PBL) activities within the mathematics classroom can positively impact students' interest in mathematics and foster their ability to identify the connection between mathematical content knowledge and creative thinking. The purpose of this workshop is to demonstrate how to implement STEM PBL strategies and activities. Discussions include the advantages of using this pedagogical strategy in mathematics classrooms. Participants explore the engineering design process through two mini-STEM PBL activities. At the end of the workshop, participants should feel confident in their ability to implement effective STEM PBL activities that foster mathematical creative thinking.*

INTRODUCTION

Mathematics is the foundation for science, technology, and engineering, but as an individual discipline, it is often perceived as simply adding, subtracting, and multiplying, etc. Mathematics is not often thought of as a creative subject (Valenti, Masnick, Cox, & Osman, 2016) that allows for exploratory opportunities in the classroom. In a traditional mathematics classroom, students are primarily taught mathematical operations; however, they also require instruction that fosters their mathematical creative thinking to be able to attain and apply advanced understanding of science, technology, engineering, and mathematics (STEM) fields (Leikin & Pitta-Pantazi, 2013). Students interested in pursuing a career in STEM fields need to be competitive and marketable by utilizing their creative thinking to come up with the next revolutionary innovation. This includes using creativity in mathematics, which translates into the other three STEM disciplines.

One method for integrating all STEM disciplines together, while promoting creative thinking, is the use of STEM project-based learning (PBL) activities (Madden et al., 2013; Oner, Nite, Capraro, & Capraro, 2016). The implementation of STEM PBL activities in the classroom allows students to explore their own ideas and test for accuracy. This process encourages students to be creative and to develop solutions to a problem that are unique from those of their peers. This is the innovative thinking that employers in competitive STEM professions are seeking.

Educators should encourage creative thinking in all disciplines; however, there is a particular need to develop ways to integrate creativity within mathematics to counter the common perception that the two entities are unrelated. This workshop was designed to demonstrate strategies for implementing STEM PBL activities in all STEM disciplines that allow for creativity, but it specifically emphasizes creative mathematical thinking design.

PROJECT-BASED LEARNING FOSTERING CREATIVITY

Problem-solving and critical-thinking skills are becoming essential for students with the increase in the number STEM careers. One possible way of motivating students to improve these skills is through creativity (Treffinger, 2007). Creativity, in our session, can be defined as an ability to design original ideas within a task or project and to come up with a variety of novel solutions (Csikszentmihalyi, 1999; McIntyre & McIntyre, 2007; Sriraman, 2004). "When we construct understandings, produce a plan of action, generate an alternative interpretation, understand an event, solve a problem, and even devise a lie to avoid trouble", our creativity is being enacted (Newton & Newton, 2010, p.112). Creativity affords students ways to explore a variety of topics and problems. In order to be considered creative, an idea does not have to be entirely novel. Small new ideas can be considered creative, as can adaptations of current ideas. If students ideate more flexibly when solving problems, they will generally be more efficient problem solvers (Star, Riddle-Johnson, & Durkin, 2016) and will typically be more successful in completing STEM majors (Pinasa, Siripun, & Yuenyong, 2017). Research supports strong problem-solving skills as essential for obtaining a STEM-related career because one must be able to think about various strategies to be able to design innovative and effective solutions (Bicer et al., 2018). In the process of identifying viable and creative solutions, it is sometimes necessary to make mistakes in order to discover different methods for solving specific problems. It is critical in STEM disciplines to explore multiple methods of approaching problems. In order to be successful in STEM careers, one must think about problems, explore multiple methods of solving specific problems, and develop multiple solutions for a problem until a final solution is found (Land, 2013). Using creative thinking helps to improve problem-solving and critical- thinking skills, leading to increasing interest in STEM careers.

GOALS OF THE WORKSHOP

The main goal of this workshop is to encourage educators and researchers to implement STEM PBL activities in order to foster students' mathematical creative thinking. Specific goals include the following:

- Define STEM PBL
- List the benefits of STEM PBL
- Explore the steps of the engineering design process
- Engage in the engineering design process
- Complete two mini-STEM PBL activities
- Make connections between STEM PBLs, mathematical content knowledge, and creative thinking
- Promote confidence in implementing a STEM PBL activity

ENGAGEMENT OF PARTICIPANTS

First, presenters explain what STEM PBL activities are and provide the benefits of using these in the classroom. Discussion related to the advantages of implementing STEM PBL activities into mathematics classrooms is encouraged. Participants then complete "The Challenge". Within this challenge, groups use various creative geometric constructions and assembly techniques to create the tallest self-standing structure. Constraints involving time and materials are presented. Participants are then guided through the mini-PBL activity and are tasked with constructing the tallest tower made from spaghetti, tape, string, and a marshmallow. At the end of the mini-PBL, presenters and participants reflect on the activity and the processes they used to create their self-standing structure.

The presenters then share the engineering design process (systematic approach followed when et adeveloping a solution for a problem with a well-defined outcome) and explain how this process helps students begin to see the connection between mathematics and creative thinking. Each non-linear step in the engineering design process (identify problems and constraints, research, ideate, analyze ideas, build, test and refine, and communicate and reflect) is discussed thoroughly with the participants, ensuring the connection to mathematics content and creative thinking. In order to introduce participants to the complexities of these steps, they are then engaged in the following mini-STEM PBL: “Drinking Water: Is it safe?”. We discuss the qualities of water that make it possible to drink and swim in and explain how to test water using a laboratory analysis. Participants are required to use their prior mathematical content knowledge and creative thinking to find the solution to the the following problem: “Make dilutions with water and food coloring and model: 1 part per 100, per 1,000, per 10,000, per 100,000, and per 1,000,000”. A reflection then takes place in which the participants answer three questions: “How did you develop your dilutions?”, “What worked?”, “What did not work?”. Lastly, during the workshop, we provide groups of participants four different PBL activities (Sink & Float: Buoyancy, The Best Cereal Box, Modeling Clay, and Aquaponics) from the book: *A Companion to Interdisciplinary STEM Project-Based Learning: For Educators by Educators* (Capraro, Whitfield, Etchells, & Capraro, 2016) and ask them to explore and become experts in their PBL. Using the jigsaw method, participants can share their expertise with other groups. Finally, discussions about how STEM PBL activities can be implemented in their specific areas are addressed.

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WORKSHOP: VARIATIONS IN OPEN PROBLEM FIELDS AS A TOOL FOR MATHEMATICAL EDUCATION: FROM BASICS TO OPEN QUESTIONS IN 90 MINUTES.

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[1]Technical University of Berlin, Germany, [2]University of Hamburg, Germany

Abstract. *This workshop aims to give insight in how to design a research-oriented experience for mathematically gifted pupils in high school. In order to do so we reflect on the creative challenge to ask interesting questions and develop adequate notions for a fruitful research field. The example to which we mainly refer considers tilings of the plain. We offer materials for possibly five days, which only partially build up on each other, for a project week within a high school. We expect the participants of this workshop to work with our work sheets from a student's and a teacher's point of view. Furthermore, we want to discuss the general idea behind such material in total. As a key feature, this includes a text-driven approach for variations of mathematical problems.*

Key words: Open problem fields, design of work sheets, tilings, Penrose tilings, text-driven variations, creativity, research-oriented experience

CONTENT OF THE WORKSHOP

This workshop has the following two main aims:

- introducing the mathematical area of tilings of the plane as a prime example of an area which is easy to understand and quickly leads to open problems or even to the edge of current research, and making use of this area in enrichment courses for mathematically gifted youth;
- a more general reflection on the question of how to find and design a similar, research-oriented field of study and corresponding work sheets as presented in the workshop. This should yield to insights on the actual mathematical research practice.

While mathematics on a research level is a creative endeavor, this often translates only partly into enrichment programs. Creativity is crucial for solving problems, but similarly important while judging which definitions and concepts are worth studying and which questions, we should try to answer in the first place. We want to communicate this openness to pupils and students by integrating it into their mathematical education.

This creative component is often linked to the picture of *geniuses*. We think this nimbus is counterproductive. The creative component should rather be framed as something, everybody can learn, and which allows to shape a mathematical topic accordingly to one's preferences.

First, we discuss how simple syntactical variations of statements can lead to new propositions to study. We shall show in how far this mechanism can be used in mathematical education to develop a more open, i.e. research-oriented experience for participating students. Hereby we want to investigate in how far we can understand (or rationally reconstruct) how mathematical theory building can be analysed on a text-level approach.

This is apparently only a first approximation concerning the heuristics actually used in mathematical practice, but yields already useful insights. Apparently not all such variations yield to fruitful fields of study and several of them are most likely not even meaningful. We develop a quasi-evolutionary account to explain why this variational approach can help to develop an understanding how new definitions replace older ones and how mathematicians choose axiomatisations and theories to study.

Making use of this, we explain how teachers could work within such an open research-oriented framework in general. Maybe against initial expectations, this does not necessarily involve an increased effort in preparation. As an example, we present five designed worksheets, which can be used consecutively, but do not have to. The purpose of these worksheets is to show the participating students how it feels to ‘navigate in unknown territory’.

We shall give a case study within the subject of ‘tilings’. There we begin with the basic question which regular (convex) polygon can be used to construct a tilling of the plane; a question in principle accessible with high school mathematics. Small variations of this problem quickly lead to new sensible fields of study. For example, allowing the combination of different regular (convex) polygons yields to Archimedean tilings of the plane, or introducing the notion of ‘periodicity’ paves the way for questions related to so-called Penrose tilings. It is easy to get from a high school problem to open mathematical research by only introducing a few notions and syntactical variations of proposed questions.

This meanly mathematical problem field can be enriched by artistic, crafting, historical and cultural discourses. This should make it possible to use this problem field not only within formats singling out especially gifted youth but as a field worth doing within the whole class. One concrete possibility for this might be via the art of Escher, who has partially already found his way into the German school systems. Other detours might be considering Islamic architecture or building and designing (f.i. Escher-like) tilings.

The material can be presented in different styles. We offer materials for possibly five days, which only partly build up on each other. This could for instance be used within a project week in high school. Shorter versions of each such project-day can be used for short working-spans less than 90 minutes. When considering the example worksheets, we do not want to focus on getting to ‘the solution’ of the problems on the worksheets, but rather on various solutions, follow-up questions and the different ways how students and teachers might interact with the proposed material.

Depending on whether there is time left, we show that this model can also be applied to newly emerging fields of mathematical research.

This workshop is based on work used for enrichment programs for mathematically gifted children and on observations from working mathematicians.

AN INTRODUCTORY WORKSHOP TO THE VISUALIZATION OF THE FIRST STEPS OF NUMBER THEORY FOR ELEMENTARY SCHOOL CHILDREN – A PYTHAGOREAN APPROACH

Peter Koehler
The Nueva School, U.S.A.

Abstract*: In this workshop we will create a mathematically and visually stimulating hands-on environment in which participants will discover and create polygonal numbers using interlinking coloured blocks. The approach is inspired by the teachings of Pythagoras and his followers, who used pebbles to turn number sequences into geometrical patterns. The interplay between numbers and geometry will give rise to questions and potential answers which can be transferable to the participants' own students in a similar setting.*

Key words: Discovery Learning, Hands-on, Odd and Even Numbers, Visualization of Number Theory

RATIONALE

Number theory is the exploration of the properties of numbers. When we arrange whole numbers with manipulatives, the properties of numbers and the geometrical relations between them become visual. This aspect of numbers can naturally be accessed by a child irrespective of their previous mathematical experience or approach to numbers. It allows for an originality of creativity, which a pure paper and pencil setting would not afford.

GOALS OF THE WORKSHOP

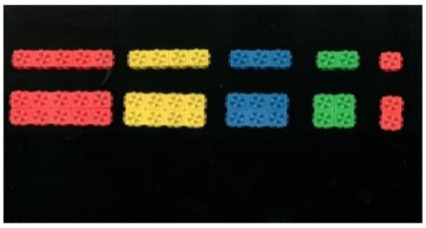
To give participants a deeper understanding of number; to inspire students to pause, reflect, try out their ideas, find out if they work and if so experiment to find out why, and formulate their findings in mathematical language; to spark mathematical creativity; to generalize; to motivate students to see the joy and beauty in math.

METHOD

We will use interlinking coloured blocks and follow a few simple rules to make patterns, shapes and numbers. We can hold shapes in our hands, rotate them, break them apart and rearrange them. As we do so we will learn and practise arithmetical concepts and procedures like area, volume, addition, multiplication, subtraction and division.

Our rules are: start off at the smallest size; go up step by step in sequential order; each term in the sequence must be a visual replica of the previous term, only bigger; if you rearrange the set of designs use the same number of blocks.

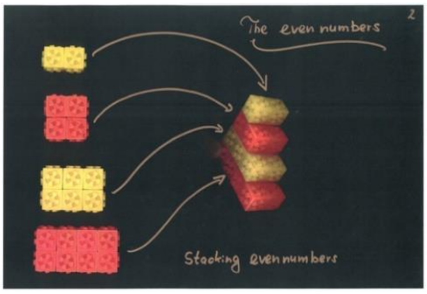
We begin by visualizing counting numbers as line numbers:



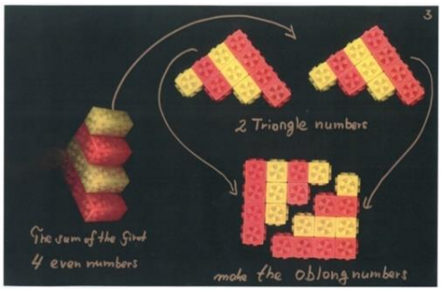
top row 1st 5 consecutive line numbers n=1,2,3,4,5.

2nd row 1st 5 even numbers 2n= 2,4,6,8,10

We distinguish odd from even numbers by giving them distinctly different geometric shapes.

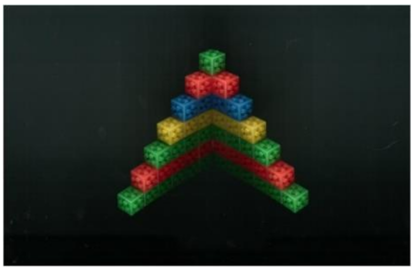


We stack the even numbers together to form a regular staircase



We split the staircase through the middle into identical triangle numbers. We join them together to form an oblong rectangle. The sum of the 1st consecutive even numbers makes twice the same triangle number or an oblong number. We can add the terms of sequences of even numbers and list the summations. The numerical pattern suggests a formula expressed with the letter n as twice a triangle number of $2T(n) = n(n+1)$

We can stack consecutive odd numbers. Summation 1+3+5+7+9+11+13 = 7x7



When we detach the two perpendicular triangular walls and rejoin them we get a 7x7 square.

Not only does a square number look like a square, and a triangle number resemble a triangle, but a pentagon number resembles a pentagon as we see from the 5th pentagonal number 1+4+7+10+13+16 = 51 below

Its internal structure is composed from the sum of the 6th square number plus the 5th triangle number.

We can put line numbers together to make triangle numbers. We can put triangle numbers together to make squares. We can experiment with squares, turning a 5 x 5 square into a 6 x 6 square. We can make a whole family of shapes in sequential order. The geometrical shapes which correspond to the numbers display their meanings more vividly than numbers alone.

These are only a very few of the ideas which can be incorporated into a hands-on visual approach to number theory for elementary school children. The field is very rich and deep and affords countless possibilities, limited only by imagination and enthusiasm.

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SHAPE UP: PROVEN SPATIAL ACTIVITIES FOR ELEMENTARY STUDENTS

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Abstract. *Spatial ability is a critical, but often overlooked component of success in several STEAM careers that has its beginnings at an early age. In this workshop, we will briefly explore research on the role that spatial thinking plays in educational and occupational STEAM innovation and expertise, including the fact that spatial reasoning is not hard-wired and can improve with practice. Participants will then engage in proven spatial investigations and activities from the National Science Foundation-funded Project M²: Mentoring Young Mathematicians for students in kindergarten through second grade and the US Department of Education Javits-funded Project M³: Mentoring Mathematical Minds for grades three through six students.*

Key words: spatial ability, mathematics, geometry, exceptional promise, STEAM

OVERVIEW

Ever since Plato's Academy for which knowledge of geometry was an entrance requirement, we have recognized the importance of spatial reasoning - the ability to physically and mentally generate, locate, represent, move and transform visual images and structures and think logically about relationships among them. Many of history's most important discoveries were due to spatial thinking and virtually every science, technology, engineering, art and mathematics (STEAM) field requires this ability to reason about and represent spatial concepts. This ability must be developed beginning with young children to foster future talent in STEAM fields. As noted over fifty years ago, "At the present time, there is a developing educational crisis, because of the unsatisfied demand for personnel trained and qualified in all fields in which spatial ability is of fundamental importance. The technical revolution has put a premium on spatial ability at all levels, whether required for tile-laying or topology." (Smith, 1964, p. 299). Longitudinal research since Smith's book was published also supports the critical importance of spatial ability. In a fifty-year follow-up study of mathematically precocious youth, spatial thinking has been found to be an important predictor of achievement in STEM disciplines (Wai, Lubinski & Benbow, 2009).

In spite of the acknowledged importance of spatial reasoning, it is an oft-neglected aspect of our curriculum. It is a critical component of mathematics that goes far beyond naming geometric shapes in elementary school or memorizing geometric proofs in high school. If mathematics is defined as the study of patterns, recognizing, creating, extending and generalizing spatial patterns in one-, two-, and three-dimensions and beyond is a major part of spatial reasoning and of mathematics in general.

It is a misconception to think that spatial thinking skills are fixed. Like other mental abilities, spatial reasoning can improve with proper experiences and practice (Uttal et al., 2013). In addition, some studies have shown that improvements in spatial reasoning can transfer to other mathematical areas (Cheng & Mix, 2012; Verdine et al., 2014). Schools play an important role in improving these skills, especially for girls (Casey et al., 2008) and students from low SES neighborhoods (Hawes, Tepylo & Moss, 2015).

WORKSHOP ACTIVITIES

In this workshop, after briefly summarizing some of this data, we will explore proven, research-based spatial reasoning activities from the National Science Foundation-funded *Project M²: Mentoring Young Mathematicians* (www.projectm2.org) for students in kindergarten through second grade and the U. S. Department of Education Javits-funded *Project M³: Mentoring Mathematical Minds* (www.projectm3.org) for grades three through six students. Both programs showed consistent, significant gains over comparison groups on all units and standardized tests for all groups as well as significant gains with large effect sizes on open response assessments. (Gavin et al, 2013; Gavin et al, 2009)

Participants will be actively involved in investigations, beginning with *Look, Make, Fix,* a tangram activity from one of the *Project M²* kindergarten units. Recognizing that number sense is also dependent on spatial reasoning, one-dimensional spatial thinking activities will include a variety of number line activities from both *Project M²* and *Project M³*, progressing from simple first grade number concepts through fractions, decimals, and large numbers for older students. Acknowledging the importance of enjoying mathematical reasoning, games will also be played, including *Triple Play* from *Project M³,* reinforcing concepts and relationships among different types of quadrilaterals. Student journal writing from several of these activities will also be shared.

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SPIROGRAPH – A TOY AS A MATHEMATICAL PROBLEM

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Abstract. *The Spirograph is a toy that consists of plastic rings with gear teeth and plastic gear wheels which allow to draw several kinds of hypotrochoid that first of all are just beautiful. From the mathematical point of view occur several questions that can be dealt with by gifted students on different levels – starting at grade six up to university. Selected aspects are shown here.*

Key words: Spirograph, Problem Solving, Hypotrochoid .

THE TOY

A typical drawing set contains of two or more plastic rings with gear teeth – the number of teeth is printed on the rings – and about 15 circles with many tiny holes to put a pen in. After fixing one of the rings on a piece of paper one places a circle in the ring, puts a pen in a hole and starts to draw rolling the circle up inside the ring. This way a curve is generated and after a while one ends up at the starting point. The complexity of the curve varies using different rings and circles.

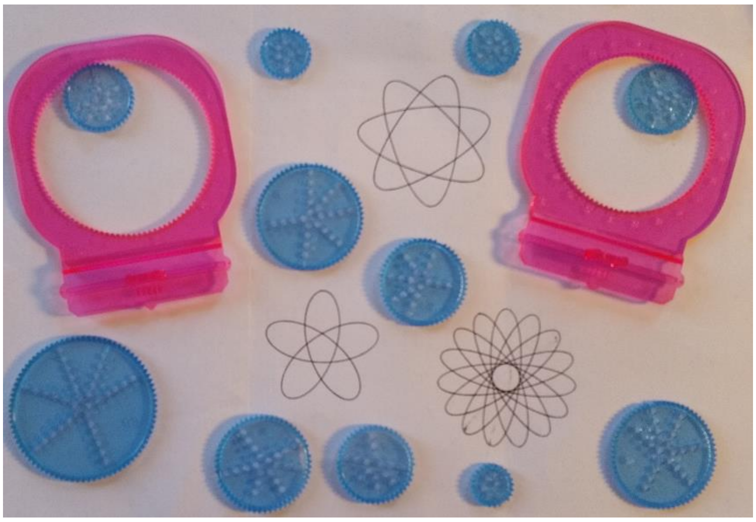

Figure 1: A spirograph toy set

THE INITIAL PROBLEM

Somebody draws several curves using different rings and circles. After producing a series of different curves one selected curve e. g. one of the curves shown in figure 1 shall be drawn again – but how was it made?

From a mathematical point of view there are three variables: the ring, the circle and the drawing hole that was used. Is it possible to reconstruct these information from the drawn curve? The answer is yes! But it needs a deep understanding of the construction process of the curve.

ORGANIZING THE MATERIAL

As in many problem-solving processes it's a good idea to start trying out systematically varying only one of the variables at a time. Figure 2 shows different curves using the same ring (with 105 teeth) and circle (with 63 teeth) but different holes.

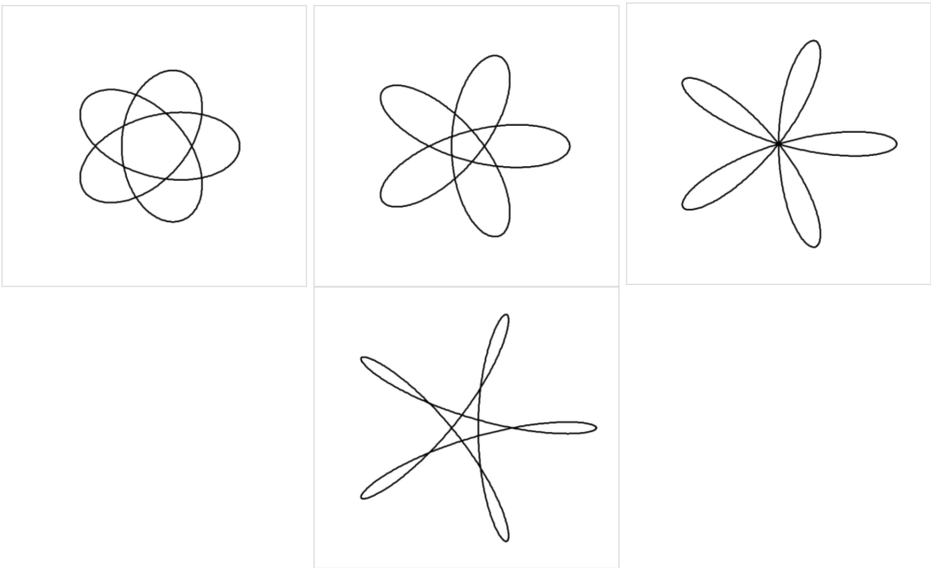

Figure 2: Different hypotrochoids drawn with the same ring and circle (made with a computer)

One similarity of these curves is obvious: each curve consists of five bows. Morphing the whole more to the border of the circle (from left to right) makes the bows slender, but in a certain way the shape stays the same. The five bows seem to be a characteristic of the used ring and circle.

The next step might be using the same ring but different circles. Now one shouldn't vary the whole for the pen – but what does that mean for different circles? One choice is to use the whole that is in the middle between the center and the border of the circle.

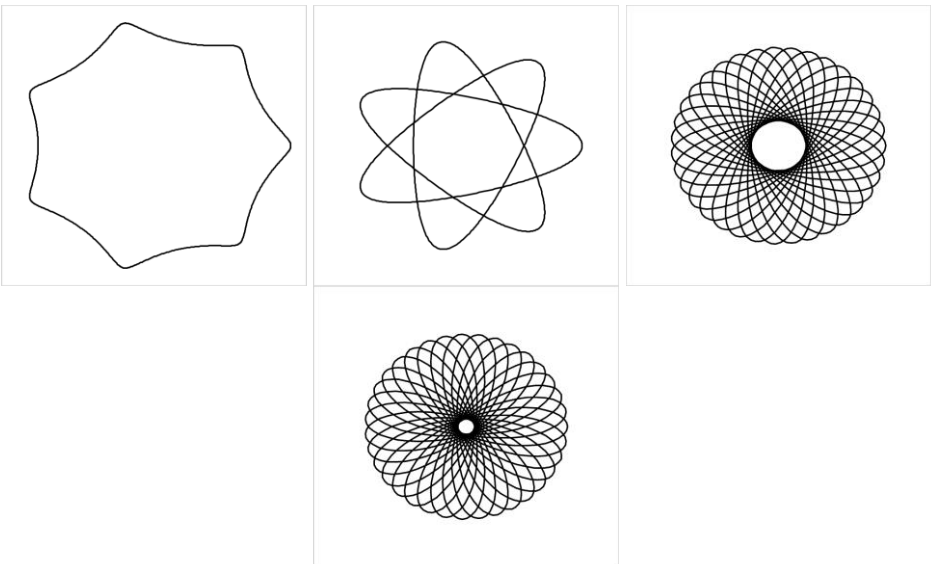

Figure 3: Different hypotrochoids drawn with the same ring (105) but different circles (15, 45, 57, 66) (made with a computer)

The selection of curves in figure 3 may lead to the hypothesis "the bigger the circle the more complicated the curve" – but this is wrong: if one inserts a curve from figure 2 in figure 3 the pattern is gone. Drawing the right curves in figure 3 takes quite long and with waning patience one may ask "when does this comes to an end?" - the answer to this question will lead to more insight in the structure of the hypotrochoids.

UNDERSTANDING THE PROCESS

This approach inverts the direction of thinking: in the initial problem one starts with the realized hypotrochoid and asks for the drawing material used. Now one starts with well known material and looks for the emerging curve. This question is much easier, and the answer will be very helpful for the initial question. In certain groups it might be a good idea to start with this question.

For analyzing the process, one draws the hypotrochoid and stops at each bow to realize what is going on. Again, the ring with 105 teeth and the circle with 63 teeth is used.

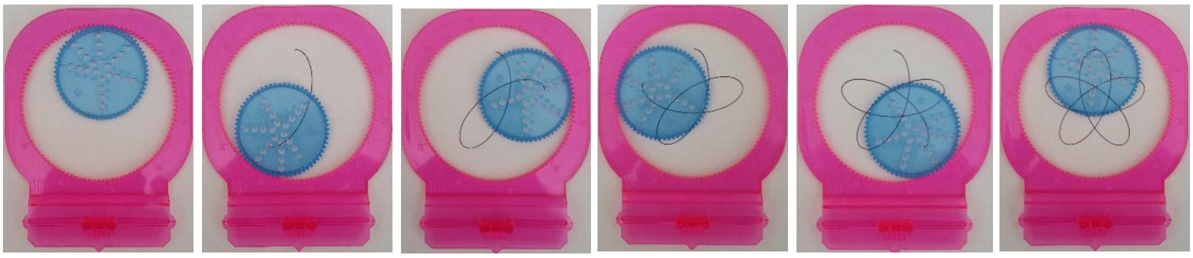

Figure 4: Drawing step by step - one step is a turn of the circle

The drawing in figure 4 shows the meaning of the number of five bows that were mentioned in the first inspection of the situation: each time the circle turns around one time one bow is drawn and after five turns of the circle the hypotrochoid is finished. Why just after five turns? The answer of this question appears after looking at this process from a different point of view:

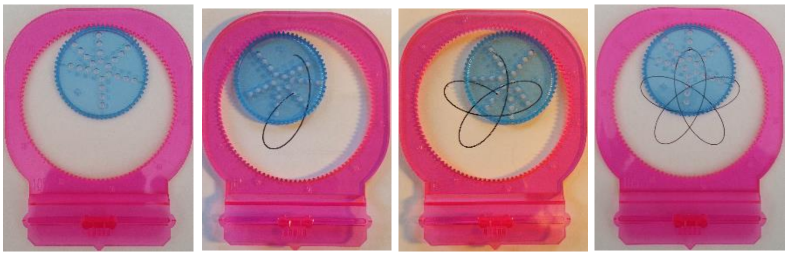

Figure 5: Drawing step by step - one step is a turn of the ring

Stopping the drawing when the circle is near the start position again shows that it needs three turns around the ring (loops) to finish the hypotrochoid.

Now there are two ways to count the teeth that roll on each other: the circle with 63 teeth needs fives turns and the ring with 105 teeth needs 3 turns. Both have to be the same:

$$5 \cdot 63 = 3 \cdot 105$$

A different representation that sorts the date of the material on one side helps more:

$$\frac{63}{105} = \frac{3}{5} = \frac{k}{n}$$

Formula 1: Relation of loops and bows equals the relation of circle and ring

If one knows the number of teeth of circle and ring one can write the numbers as a fraction an simplify the fraction. Then one knows the denominator is the number of bows (short name *n*) of the hypotrochoid and the nominator (short name *k*) is the number of loops.

BACK TO THE INITIAL QUESTION

What is discovered up to now:

The number of bows is the denominator of a fraction. This can easily be taken from a realized hypotrochoid. Then one needs the nominator – this must be the next step to discover in the hypotrochoid. Once nominator and denominator are known, the fraction must be expanded in a way that numbers of available rings and circles occur. This last step is easy. So, one question is left:

How to get the nominator from the realized hypotrochoid?

In figure 5 one can discover that in each loop the curve is drawn one time around the midpoint of the hypotrochoid! It is like drawing circles around the midpoint that just don't fit perfect. The number of these circles / loops is the nominator! Counting these loops is easy: one starts at the center of the hypotrochoid, moves outside and counts the lines crossing on this way – it's just counting loops around the center.

LOOKING BACK

The key to solve the problem was to change the direction of thinking: the initial question started with the realized hypotrochoid and aimed for the circle and ring – the solution was achieved using a known circle and ring and analyzing the process.

Analyzing the process succeeded by looking for some kind of elementary object: drawing step by step produced a bow in the first step that was repeatedly drawn in the further steps. This concept uses symmetry in the way Pólya (1973) described it. Pólya emphasized that symmetry is not only meant in the usual geometric meaning but also in a general, logical meaning: Symmetry, in a general sense, is important for our subject.

> If a problem is symmetric in some ways, we may derive some profit from noticing its interchangeable parts and it often pays to treat those parts which play the same role in the same fashion. (Pólya, p. 199, 1973)

In figure 4 the elementary bow is drawn five times in the same way, rotated by 360°/5.

In figure 5 the symmetry used is less evident: each loop is one third of the whole curve, but as the starting point of the loop is not always the same the loops look different. The first and the third loop are similar: one is the mirror picture of the other.

Looking back also means, one has to control whether the answer is complete: we now can identify the ring and the circle - but not position of the hole. Practically one would try this out. But can the location of the hole be calculated some way using the drawn hypotrochoid?

Looking on figure 2 shows that the hypotrochoid has different sizes. To derive advantage from this concept one must describe more exactly, what "size" means. One way is to look at the circumcircle and the incircle of the hypotrochoid: they both change when the drawing whole is changed.

The radius of the circumcircle (r_c) and the incircle (r_i) of the hypotrochoid can be measured as well as the radius of the circle (r_1) and the ring (r_2) as one already knows, which ring and which circle one has to use. In figure 6 a ring and a circle (in two positions) are drawn in extreme position. The circumcircle and the incircle are drawn as pointed lines. The distance of the hole to the center of the circle is named pr_1 (with p the percentage of r_1 that identifies the distance of the hole from the midpoint of the circle). Now two equations can be derived:

$$r_a = r_1 - r_2 + p \cdot r_2$$
$$r_i = r_1 - r_2 - p \cdot r_2$$

Formula 2: Calculating the position of the hole

Either of this equations may be used to get pr_1 or p to identify the hole in the circle, that was used drawing a given hypotrochoid. If this equations are formulated with real numbers instead of variables they are very simple to solve.

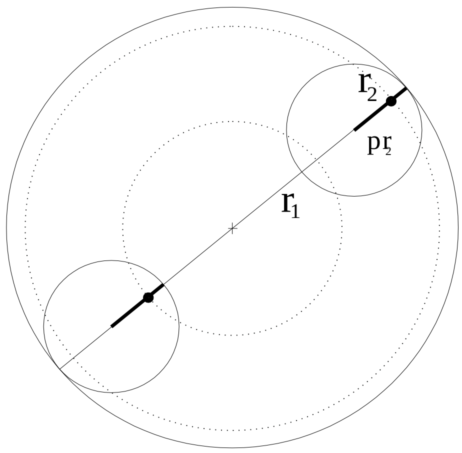

Figure 6: Size of the hypotrochoid

Checking the result again brings another missing aspect into focus: if one wants to count n in figure 2 this works with the two left hypotrochoid but not with the right ones! There are two types of hypotrochoids: for type 1 (left ones) the initial problem is solved, for type 2 more work has to be done.

BACK TO THE PROBLEM

The obvious issue counting the circles around the midpoint of the hypotrochoid is that the counting leads to different result depending on the direction of counting: on a straight line from the midpoint to the right one counts four loops, but diagonally upwards one counts only two loops: but we already know that the right number is three. The first idea might be that one has just to uses the average: this is nice but wrong.

If a concept works on parts of the problem but not on other parts one should ask oneself “why?” The answer can be found again drawing hypotrochoids from figure 2 slowly: the pen just passes the midpoint of the right hypotrochoid on the wrong side each time. To fix this one may try to mentally push all the lines of the hypotrochoid over the midpoint as if one morphs the right hypotrochoid from figure 2 to the one on the left. Doing this one has to analyze how to count. For this idea a more complex example is helpful (figure 7).

Starting at the midpoint counting to the right (this counting pass is used from now on) one crosses seven lines of the hypotrochoid. Now one calculates how this number changes when the loops are mentally pushed over the midpoint of the hypotrochoid. In figure 7 each loop is marked at one end by drawing it thicker. When one pushes the rightmost thicker marked loop over the midpoint this loop is not counted any longer so a “-1” is put at this loop. Moving counterclockwise the same applies to the next five loops (6 in total). All other loops must be counted additionally once pushed over the midpoint so they get a “+1” (15 in total). This leads to the calculation for the right number of loops:

$$n=7-6+15=16$$

Formula 3: Recalculating *n*

Now the initial problem is solved for this type of hypotrochoids too.

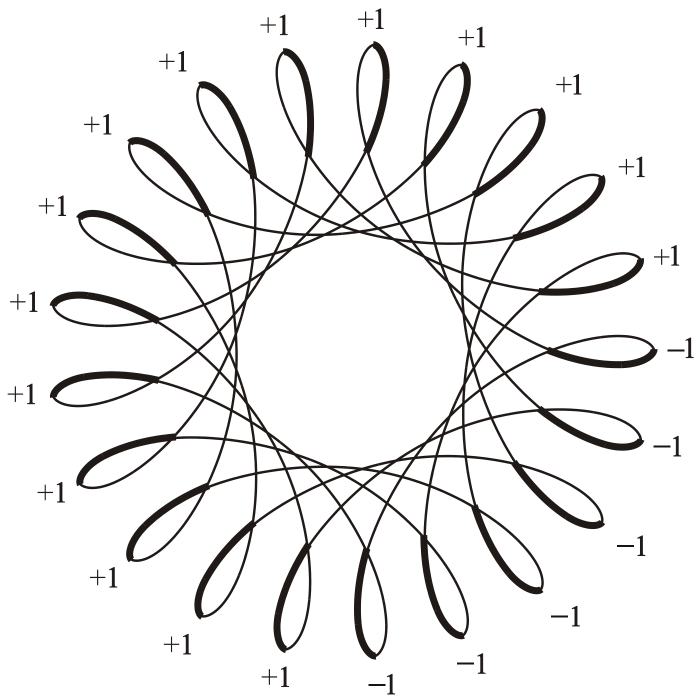

Figure 7: Recount

GOING ON

Once the initial problem is solved students of higher grades could start to produce the curves using a computer. Using a computer makes it possible to draw much more different hypotrochoids as the material of the toy-set is no longer a limiting factor. In addition, with a computer one can produce hypotrochoids that are impossible to draw using real circles for example in a program one can choose $p>1$ (p from figure 6).

The calculation can be done using a spreadsheet which means that only the mathematical expressions are used and little programming knowledge is necessary. For this activity a good understanding of functional thinking and knowledge of trigonometry is required.

Again, analyzing the drawing process in detail, now from another perspective, is essential to fill the spreadsheet step by step. At first one has to come to the decision where to put the center of the coordinate system. Obviously, this should be the symmetry center of the hypotrochoid. Like a satellite the midpoint of the circle of the toy-set moves on a circle around the midpoint of the hypotrochoid. On the other hand, the hole with the pen moves in circles around the midpoint of the toy-set-circle. That means one has to draw circles around circles, so the basic mathematical formula used is

$$x(\alpha) = r \cdot \cos(\alpha) \; y(\alpha) = r \cdot \sin(\alpha)$$

Formula 4: The root formula: a circle of radius *r*.

Figure 8 helps using this formula to create the spreadsheet. It is a helpful concept to realize the different steps of the calculation in different rows of the spreadsheet. This way the problem is naturally divided into parts following Pólya (1961, p. 129): "Divide each problem that you examine into as many parts as you can and as you need to solve them more easily."

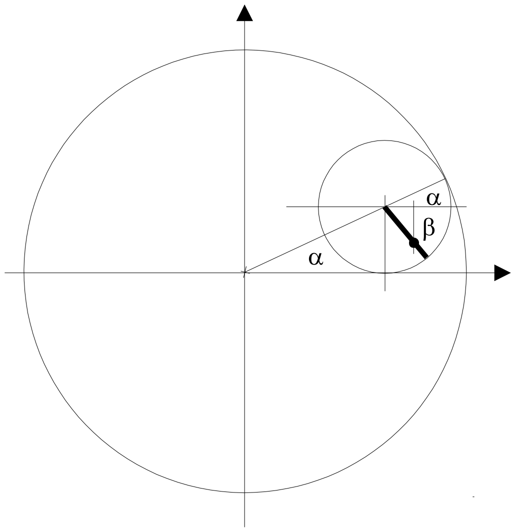

Figure 8: Creating formulas for computer use

The angel α is the independent variable starting with 0° ending at *k* times 360°. This list should be in the first row (A).

The angel β changes faster than α: the factor is *n* divided by *k.* This is the next row (B). As β changes in the opposite direction of α a "-" is necessary.

The coordinates of the midpoint of the small circle are calculated in the next two rows (C,D) using formula 4. The radius is $r = r_1 - r_2$ with r_1 and r_2 from figure 6.

In the next two rows (E,F) the distances (*x* and *y* coordinate) of the drawing point to the midpoint of the circle are calculated using formula 4 again with $r = pr_2$ and $\alpha+\beta$ instead of α.

The last step is just to add the values form row C and E (in row G) and D and F (in row H). Drawing these rows in an *x*-*y*-diagram produces the hypotrochoid.

Once the spreadsheet is finished it allows additional examinations of the spirograph as the boundaries of the toy set no longer apply. Now every ring diameter and circle diameter can be used and the drawing whole can even be outside of the circle – mathematics makes it possible.

Using a spreadsheet one can analyze variations of the situation which requires to adapt the formulas: what if its not a circle moving in a ring, but a circle moving on a circle or a straight line or something different like parabulas?

HIGHER LEVEL QUESTIONS

Using analysis to examine the curves drawn with a spirograph leads to university mathematics. On university level further questions can be posted for example how long the curve is or what can be said about the tangents of the curve. This leads to deeper questions dealt with in differential geometry with surprising mathematical and physical results – but these mathematics exceed this paper.

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SYMPOSIUM

TAPPING MATHEMATICAL CREATIVITY THROUGH PROBLEM SOLVING: PROBLEM-MATRIX FRAMEWORK FOR TEACHING FOR CREATIVITY IN THE MATH CLASSROOM

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Abstract. *Much of academic content and teachers' daily instruction practices in the math classroom could be considered as an aspect of problem solving (Bahar & Maker, 2016). To support students in developing as creative problem solvers, teachers need a systematic understanding of what a creativity and creative problem solving growth trajectory looks like in the math classroom. For these purposes, presenters offer a 'Problem Matrix Framework', which will guide mathematics teachers in cultivating creative potentials through production, fluency, originality, and detail in the solving of complex and realistic problems. Presenters will also share the findings of their study on how different types of problems tap mathematical creativity differently.*

Key words: Mathematical creativity, problem solving, problem types, open-ended problem.

DEVELOPMENT OF MATHEMETICAL CREATIVITY

How might teachers support the development of students' creative potential? This is a longstanding question in the field of creativity studies (Guilford, 1950) that has received renewed interest amongst educational policy-makers, business leaders, and government officials (Beghetto & Kaufman, 2013). Indeed, helping students develop their capacity to think creatively has long been viewed as one of the best, yet often neglected, ways to prepare students for an uncertain future (Dewey, 1933; Guilford, 1950; Vygotsky, 2004). In this symposium we will discuss the role of teacher and use of problem types (open-ended vs closed problems) on the development of creative potentials in math classrooms. The following questions will be addressed:

- How might teachers support the development of students' mathematical creative potential?
- How does the type of a math problem (open-ended vs closed problems) tap/activate mathematical creativity differently?

Much of academic content and teachers' daily instruction in math classroom could be considered as an aspect of problem solving (Bahar & Maker, 2011; Bahar & Maker, 2016). In this symposium, we will seek responses to these questions specifically from a mathematical problem solving perspective, especially by referring to the "Problem Continuum Matrix", which was developed by Maker and colleagues (Maker, 1993; Whitmore & Maker, 1985). In this context, problems were classified as either closed or open based on the number of alternatives available to the problem solver. For example, a problem was defined as closed if it could be solved in only one way and open if it could be solved in an infinite number of ways.

Type		Problem		Method		Solution	
		Presenter	Solver	Presenter	Solver	Presenter	Solver
Closed	I	Specified	Known	Known	Known	Known	Unknown
	II	Specified	Known	Known	Unknown	Known	Unknown
	III	Specified	Known	Range	Unknown	Known	Unknown
Open-Ended	IV	Specified	Known	Range	Unknown	Range	Unknown
	V	Specified	Known	Unknown	Unknown	Unknown	Unknown
	VI	Unknown	Unknown	Unknown	Unknown	Unknown	Unknown

Table 1.1 Problem Continuum

To support students in developing as creative problem solvers, teachers need a systematic understanding of what a creativity and creative problem solving growth trajectory looks like in the classroom. At this point, we believe 'Problem Continuum Framework', which will guide teachers in cultivating creative potentials through production, flexible use, originality, and addition of richness and detail in the solving of complex, realistic problems in specific academic and intellectual domains. Creativity is applicable to most situations and problems and is relevant to ill-defined or open-ended problems or those that require restructuring, overcoming fixation, or achieving insights.

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RESEARCH WITHIN THE FRAMEWORK OF THE HAMBURGER MODEL FOR THE PROMOTION OF PARTICULARLY MATHEMATICALLY GIFTED CHILDREN AND ADOLESCENTS

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Abstract: *This symposium is intended to provide insights into research concerning mathematical giftedness conducted within the William-Stern-Gesellschaft für Begabungsforschung und Begabtenförderung e.V. (Association for talent research and fostering; WSG e.V.). More specifically, results on gender differences in mathematical performance, Need for Cognition (NFC), Learning and Achievement motivation (definitions see below), as well as intelligence are presented. Moreover, this symposium aims at establishing new contacts as well as exchanging views with other organizations and programs that are concerned with talent research and fostering to enable a sustainable exchange.*

Keywords: Mathematical creativity and giftedness; Formal and informal learning

INTRODUCTION

The William-Stern-Gesellschaft für Begabungsforschung und Begabtenförderung e.V. (Association for talent research and fostering; WSG e.V.) was founded in 1985/86 as a registered non-profit association, which pursues the goal to promote education, particularly of gifted and talented children and adolescents. The symposium especially focusses on research done within the framework of the *Hamburger Model for the promotion of particularly mathematically gifted children and adolescents* (Kießwetter, 1985). This interdisciplinary program developed by mathematicians, educational scientists and psychologists in cooperation with the John-Hopkins-University (Baltimore) started in the 80s. The program aims at fostering mathematically gifted children and adolescents who are being selected by an entrance test, the so called "talent search mathematics". The talent search is comprised of two measurements of mathematical abilities: The German version of the Scholastic Aptitude Test for mathematics (GSAT-M; see Wieczerkowski, 1989), which assesses knowledge in mathematics and the "Hamburger Test für mathematische Begabung" (HTMB, Kießwetter, 1985), which examines another part of mathematical giftedness, particularly mathematical creativity. Children who pass the admission criteria participate in approximately 22 biweekly classes over the course of the schoolyear. In these classes, the students work on challenging mathematical problems. A typical session usually looks like this: At the beginning (9:30 a.m.), additional information about the problems of the previous session may be given to the students, and a new problem area is introduced by a short lecture. After this the students are free to split up into groups and start working on the problem. They are nearly free to choose their working methods and the kind of problems they want to tackle. At around 12:00 p.m. the groups gather for the final plenary session to report and compare their results. The session ends at 12:30 p.m.

Although homework is not required, the students are encouraged to continue their work on the problem areas during the weeks between two sessions. The teaching material used in the classes contains problems or problem areas from elementary mathematics (graph theory, geometry, combinatorics, number theory, game theory). Outstanding ideas of the students (e.g., proofs, computer programs, games) are written in their own working papers. In addition to the biweekly sessions, every spring there is a "math camp". This trip serves as a great opportunity to strengthen friendships and team spirit as well as a possibility to tackle even more extensive mathematical problems.

In this symposium results from the entrance tests of the talent search and their relationship to different psychological variables will be presented. Those variables are: *Mathematical ability, NFC, Learning* and *Performance motivation,* as well as *Intelligence*. In the following, brief definitions of those constructs are given. Mathematical ability was assessed with two different instruments as described above. Gender differences in the performance on these tests will be discussed. NFC is a motivational construct referring to *"an individual's tendency to engage in and enjoy effortful cognitive endeavors"* (Cacioppo, Petty, & Kao, 1984, p. 306). Results on its relationship to mathematical performance, scholastic achievement and intelligence are presented. *Intelligence* refers to general cognitive ability and is operationalized by the intelligence quotient (IQ) which was originally introduced by William Stern (1911) and further developed by David Wechsler (1944), for an overview see McGrew (2005). The other constructs, *Learning* and *Achievement motivation* refer to different types of goal orientation (Preckel, 2014). While individuals motivated by learning goals aim at developing and increasing their knowledge abilities, performance goal oriented individuals aim at demonstrating their abilities in front of others (approach-performance goals) or hiding their weaknesses (avoidance-performance goals). Further details on the modalities of the different studies being presented at the symposium can be found below.

TALENT SEARCH MATHEMATICS: DIFFERENCES IN MATHEMATICALLY GIFTEDNESS IN GIRLS AND BOYS?

Regarding gender differences in mathematical abilities, consistent but small effects are reported. The PISA consortium Germany (2004) reports better mathematical achievements of boys. The IGLU study showed that girls are better at reading, but boys are better at math and science (Schwippert, Bos, & Lankes, 2003). Indeed, in meta-analyses minimal gender differences in mathematical performances were summed up (Hyde, Fennema, & Lamon, 1990). However, effects couldn't be replicated (Hedges & Nowell, 1995). Therefore, the gender similarities hypothesis is proposed, which states that boys and girls are equally in most psychological variables (Hyde, 2005). Further evidence for this hypothesis was provided in a replication study (Zell, Krizan, & Teeter, 2015).

Talent searches in the field of mathematics have shown a ratio of more than 13 to 1 in favor of the boys in the top 2% of the participants (Benbow, 1990). Already at primary school age, this phenomenon of underrepresentation of girls occurs in projects and competitions for the promotion of talents (Benölken, 2014). Differences in means and ratio could arise through the attribution of mathematical giftedness to boys by teachers and parents. In the recent study, the data of participants of the talent searches from the years 2003-2016 were analyzed. Operationalized using the composite score of both mathematical tests (see above), the results of the best 25% of the participants (N=592, 156 girls) were explored. The mean age of this sample was 12.2 years (*SD*=0.6, *min*=9.2, *max*=14.1). *N*=347 (77.5%) participants were nominated from their teachers, 53 (11.8%) nominated themselves, and 48 (10.7%) were nominated by teachers and themselves. Overall, we found significant differences in variances depending on the measurement ($F(1, 590)=4.544$, $p=.033$, $\eta^2=.008$). Furthermore, the correlation of the Hamburger Test für mathematische Begabung (HTMB, see above) with the German version of the Scholastic Aptitude Test for mathematics (GSAT-M, see above) in this selected sample is positive but low ($r=.131$, $p=.001$). In addition, significant correlations of the HTMB only appear with the grade in mathematics ($r=-.113$, $p=.01$), not with the first foreign language or German. Furthermore, significant correlations were found for the GSAT-M with the school grades in mathematics ($r=-.173$, $p<.001$) and the first foreign language ($r=-.109$, $p=.02$), but not with the school grade in German.

Moreover, there was a significant interaction between gender and kind of test ($F(1, 590)=20.311$, $p<.001$, $\eta^2=.033$). Girls achieved higher scores on the HTMB (*M*=15.12, *SD*=3.862, *min*=6, *max*=28) compared to boys (*M*=14.57; *SD*=4.215, *min*=5, *max*=31), while boys scored higher on the GSAT-M (*M*=33.03, *SD*=6.708, *min*=9, *max*=58) than girls (*M*=30.25; *SD*=6.384, *min*=12, *max*=47). All calculated effects were small. Possible causes of the results of the recent study and implications will be discussed.

NEED FOR COGNITION: HOW IS NEED FOR COGNITION RELATED TO SCHOOL ACHIEVEMENT IN PARTICULARLY MATHEMATICALLY TALENTED YOUNG PEOPLE?

This talk presents results regarding the relationship of need for cognition (NFC), mathematical performance and scholastic achievement, as well as intelligence. NFC refers to *“an individual’s tendency to engage in and enjoy effortful cognitive endeavors“* (Cacioppo, Petty, & Kao, 1984, p. 306). It is proposed to be positively related to scholastic achievement and to incrementally predict it beyond intelligence (Preckel, 2014). The research presented aimed at replicating these findings. It includes two separate analyses of different data sets.

	whole sample *N*=427 Kendall's Tau	upper half *n*=213 Kendall's Tau	bottom half *n*=214 Kendall's Tau
German grade	.136***	.109*	.144**
Math grade	.097*	.088	.053
Foreign language grade	.100**	.133*	.071
GPA	.138***	.138**	.115*
GSAT-M	.140***	.064	.121*
HTMB	.156***	.101*	.156***

Table 1: *Results study 1: Relations of NFC and different variables per subsample.*

Note. GPA = Grade Point Average; GSAT-M = German version of the Scholastic Aptitude Test for mathematics; HTMB = "Hamburger Test für mathematische Begabung"; * p < .05, ** p < .01, *** p < .001

The first study, based on data of the participants of the talent searches from the years 2016, 2017 and 2018 (*N*=427), aimed at investigating the relationship of NFC and scholastic achievement in dependence of the performance on the entry test of the promotion program. The results of the correlation analysis of the first study can be found in Table 1. Small positive correlations between NFC and scholastic achievement assessed via current grades in German, mathematics, and foreign language classes were found. Moreover, NFC was positively correlated with the grade point average (average of the grades in the subjects named above) and mathematical performance scores. A median split according to mathematical performance scores resulted in two groups that significantly differed in NFC scores ($F(1, 425)=12.92$, $p<.001$, $\eta^2=.030$). Moreover, the relations depicted in table 1 indicate possible differences in the predictive power of NFC regarding academic achievement between the two groups. Possible reasons and implications of the findings are discussed.

	NFC		
	Whole sample N=63	Gifted sample n=37	Non-gifted sample n=26
grades	Foreign Language $\tau(59)=.237$, $p=.013$ GPA $\tau(61)=.183$, $p=.045$	Foreign Language $\tau(35)=.138$, $p=.274$ GPA $\tau(36)=.104$, $p=.390$	Foreign Language $\tau(23)=.438$, $p=.004$ GPA $\tau(24)=.315$, $p=.031$
mathematical abilities	GSAT $r(62)=.030$, $p=.815$	GSAT $r(35)=.077$, $p=.655$	GSAT $r(25)=-.017$, $p=.936$
intelligence	$r(62)=-.058$, $p=.650$	$\tau(35)=.006$, $p=.958$	$r(25)=-.228$, $p=.263$
GPA controlled for intelligence	$r(60)=.197$, $p=.124$	$r(34)=.095$, $p=.581$	$r(23)=.401$, $p=.047$

Note. NFC = Need for Cognition. Correlations between NFC and grades in German and math classes are not presented, as they were insignificant in all samples.

Table 2: *Results study 2: Relations of NFC and different variables per subsample.*

The second study, including data of participants of the promotion program in 2016 (N = 63), aimed at investigating the relationship of NFC and scholastic achievement in dependence of intelligence assessed with the Grundintelligenztest Skala 2 - Revision (CFT 20-R, Weiß, 2006). The CFT 20-R is based on a model of intelligence that comprises two distinct factors: crystallized and fluid intelligence (Cattell 1963, see Weiß, 2006). The CFT 20-R measures the general fluid ability which can be described as the ability to recognize and process figural relations and logical puzzles of different complexity within a certain time. The general fluid ability is a well establish estimate of the general intelligence (McGrew, 2005). The sample of the study was divided into subsamples according to IQ scores assessed with the CFT 20-R. This division resulted in an intellectually gifted subsample (IQ>130; n=37) and an intellectually normal to above average gifted sample (IQ≤130; n=26). The results of the correlational analysis are presented in Table 2. Small positive correlations between NFC and the grade point average (GPA, see above, comprising grades of German, mathematics, and foreign language classes) were found. There were no significant relations between NFC and scholastic achievement in the intellectually gifted subsample, whereas NFC was moderately related to the grade point average, as well as the grade in foreign language classes in an intellectually non-gifted subsample. NFC showed incremental predictive power beyond intelligence only in the latter subsample. Possible reasons and implications of these findings are discussed. Subsequent questions of interest regarding the role of NFC in mathematical giftedness will be addressed.

MATHEMATICAL TALENT AND MOTIVATION: HOW ARE MATHEMATICAL TALENT, LEARNING AND ACHIEVEMENT MOTIVATION AND NEED FOR COGNITION RELATED IN PARTICULARLY MATHEMATICALLY GIFTED YOUNG PEOPLE?

This study aims at investigating motivational and personality characteristics of mathematically gifted adolescents attending our talent search and the ensuing promotion program. During the talent search in 2017 and during the "math camp" of the promotion program, the relations between different personality and motivational facets were examined. Most of the data was acquired during the talent search in 2017. Besides scores on the GSAT-M, the participating students (N = 155) completed questionnaires to assess *Need for Cognition* (NFC; NFC-Teens, Preckel, 2016), *Learning* and *Achievement Motivation* (Skalen zur Erfassung der Lern- und Leistungsmotivation, SELLMO, Spinath et al., 2012), and provided information on their grade point average (GPA, see above). Additional data was gathered during the "math camp" 2017 (*n*= 33). The SELLMO includes four different domains of motivation: *Learning goals*, *approach-performance goals*, *avoidance-performance goals* and *work avoidance*.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1) GSAT-M							
(2) GPA	.294***						
(3) NFC	.203**	.235**					
(4) LG	.102	.286***	.541***				
(5) ApPG	-.072	-.074	-.015	.163*			
(6) AvPG	.033	-.162*	-.177*	.051	.581***		
(7) WA	.044	-.250***	-.521***	-.368***	.190**	.379***	
(8) German	.240***	.845***	.197**	.291***	-.011	-.175*	-.124
(9) Mathematics	.459***	.662***	.230**	.181*	-.114	-.187*	-.132
(10) English	.100	.809***	.09	.200**	-.078	-.171*	-.078
(11) Biology	.148*	.721***	.209**	.253***	.006	-.225**	-.129
(12) Physics	.268**	.608***	.306***	.14	-.095	-.222*	-.191*

Note. GSAT-M = German version of the Scholastic Aptitude Test for mathematics. GPA = Grade Point Average. NFC = Need for Cognition. LG= learning goals. ApPG= approach-performance goal. AvPG = avoidance-performance goal. WA= work avoidance. * p < .05, ** p < .01, *** p < .001

Table 3: *Correlations within the whole sample.*

GSAT-M scores showed the highest correlation with math grades and were not correlated with any of the SELLMO-scales. NFC was positively correlated with learning goals and negatively correlated with work avoidance. Correlations calculated within the whole sample of this study can be found in Table 3, with the exception of the intercorrelation of school grades. Regression analyses revealed a significant influence of NFC, math grade and work avoidance on GSAT-M scores. In the context of this talk, the descriptive results as well as the connections between these motivational and cognitive measures are presented and discussed, depending on the respective sample.

MATHEMATICALLY GIFTEDNESS AND GENERAL INTELLIGENCE: HOW IS GENERAL INTELLIGENCE RELATED TO MATHEMATICALLY GIFTEDNESS?

During the math camp 2018, the attendees of our promotion program participated in a validation study of the German version of the Wechsler Intelligence Scales for Children – Fifth Edition (WISC-V; Petermann, 2017). Altogether, 35 students (23 boys) at the age of 13 to 15 were examined. The WISC-V is a broad intelligence test based on the CHC model of cognitive abilities (WISC-V; Petermann, 2017). The CHC model combines different theories of intelligence (e.g. Thurstone, Cattell, Carroll), into one model (McGrew, 2005). In this model, broad cognitive factors combine differentiated narrow domains of ability. The broad factors of the WISC-V can be summarized in one overall factor (Full Scale IQ; FSIQ). The FSIQ reflects a child's general intellectual ability. In addition, five primary index scores are provided that make up the intelligence profile: the Verbal Comprehension Index (VCI), the Visual Spatial Index (VSI), the Working Memory Index (WMI), the Fluid Reasoning Index (FRI), and the Processing Speed Index (PSI). Moreover, there are also five ancillary index scores providing more information for special groups like highly talented students. In particular, the General Ability Index (GAI) and the Cognitive Proficiency Index (CPI) can be used. The GAI consists of subtests from the verbal comprehension, visual spatial, and fluid reasoning domains, while the CPI comprises working memory and processing speed measures. The GAI subtests are less affected by the capabilities of working memory and processing speed (Daseking, Petermann, & Waldmann, 2008). It has already been shown for the previous (fourth) edition of the Wechsler Intelligence Scale for Children (WISC-IV; Petermann, & Petermann, 2011) that the difference between these cognitive abilities varies with overall cognitive ability level (Daseking, Petermann, & Waldmann, 2008). There is evidence regarding a substantial proportion of children with high intellectual ability that tend to have a lower CPI than GAI (Petermann, 2017). In addition to the primary subtests, the arithmetic subtest was also administered. Previous research within a project to promote mathematically gifted children in primary school age has found only small correlations between performance at the mathematical entry test and intelligence (Nolte, 2012), raising the question how strongly intelligence and mathematically talented are related. The present results indicate that the group of the mathematically talented children have also high scores in general intellectual ability. The average of the general intelligence (FSIQ) was 132.7 and ranged from 119 to 150. The purpose of this talk is to take a closer look at the intelligence profiles and present the implications for future research.

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POSTER

AFFECTS OF MATHEMATICALLY GIFTED STUDENTS RELATED TO REVOLVING DOOR MODELS

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INTRODUCTION

Mathematics is not seen to be an 'authority', it is an attitude by asking 'why' (Freudenthal, 1982). Such an attitude is represented by Florian, a sixth-grade student, in his behavior during mathematics lessons. Because of his excellent achievements and both his curiosity and interest in mathematically complex phenomena, he left one mathematics lesson per week for a half-year, and worked independently on a freely chosen project (in his case the programming of a robot). During the presentation of his project he reported about his joyful working and his motivation to find a solution for every arising problem. Florian is an example for mathematically gifted students who are participating in a revolving door model. In Germany, such models usually are educational school programs, mostly based on the enrichment programs of Renzulli that provide "the mechanism for students to come into and out of advanced levels of task-specific enrichment as the need arises." (Renzulli, Reis & Smith, 1981, p. 5). They leave regular lessons in order to deal with individual contents. Florian's description of joy of both learning and willingness to make an effort arises some important research questions: Which variables of revolving door models have which kind of (e.g. motivational or volitional) impacts on the emergence of a mathematically gifted student's potential(?) and how can such impacts be characterized? This research interest is pursued in a dissertation project, which is currently in its initial stage. The present article will give a first overview on this project by outlining its theoretical background, aims, design and first results from an exploratory questionnaire study.

THEORETICAL BACKGROUND

Pursuant to the article 26 of the UNESCO definitions of the rights of children "all children – including intellectually gifted ones – have a right to an education that will foster the development of their abilities and personality to their fullest" (Gagné, 2004, p. 169), which is sometimes realized by implementing revolving door models. The dynamic and complex idea of giftedness as a product of creativity, abilities and task commitment of Renzulli (1978) is the theoretical base of popular revolving door models, especially of those implemented variously at German schools – e.g. participating in higher classes or working independently on freely chosen projects. These enrichment programs focus on individual processes of learning including individual contents and also individual methods and strategies of learning for each student (Renzulli, 1976). This approach is attended by the very different characteristics of giftedness and learning needs of capable pupils. Several investigations of mathematics education give examples for mathematically gifted pupils with special needs, such as difficulties in reading and spelling (Nolte, 2018; Fischer & Käpnick, 2015). On the one hand, current modellings of mathematics education of mathematical giftedness correspond to Renzulli's approach concerning the connecting of dynamic, holistic and area-specific perspectives, but on the other hand, they are more differentiated (Fuchs & Käpnick, 2009). This provides the foundation of the investigation outlined in this article. According to Greiten (2016), both the specific connection of such approaches as a basis of revolving door models and research as to such models in mathematical contexts in general have to be seen as current desiderata. Current research only refers to testimonials from schools participating in their own special concept of a revolving door model (e.g. Peters, 2008; Rogolla, 2009).

OUTLINES OF BOTH AIMS AND THE INVESTIGATION'S DESIGN

Based on this current research status, the main purposes of the study are

(1) to clarify the term 'revolving door model' from a perspective of mathematics education, and to deduce which variables of such models have which kind of impacts on the student's development of personality and gifted potential by theoretical-analytical studies.

(2) to investigate the impacts on participants empirically, in particular with regard to mathematical giftedness traits and personality-supporting attributes.

(3) to derive practical consequences for the individual support of mathematically gifted students.

To answer these research questions, the project is divided into three sub-studies. (1) A questionnaire study will be carried out to mathematics teacher at schools participating in a revolving door model in order to explore the potential impact. (2) An interview study with teacher and children will be conducted to determine impressions of the first-mentioned study in deeper way. (3) About 20 case-studies on students taking part in revolving door models will be conducted (synthesizing various tools such as indicator tests, interviews with students, adults and teachers, and rating sheets of participating class observation) in order to gain differentiated insight into impacts on gifted student's development of personality and potential as well as in order to explore possible typifications.

FIRST RESULTS FROM THE QUESTIONNAIRE STUDY

Because of the explorative character of the questionnaire study a qualitative design was chosen. Apart from collecting some personal data the questionnaire consists mainly of the open question "How does participation in a revolving door model affect students?" (translated from German) and is supported by some guidance questions (for example, "How does participation in a revolving door model affect student's behavior in regular classes?"). Mathematics teachers from primary and secondary schools participating in a revolving door model completed the questionnaires in written form. So far, about 25 questionnaires have been evaluated, applying a qualitative content analysis (Mayring, 2010). The formation of inductive categories was adapted to a theoretical framework based on modellings of mathematical giftedness. First evaluations of the questionnaires indicate that taking part in a revolving door model has observable impacts on the student's behavior in both the revolving door lessons and the regular classes, especially on cognitive variables, aspects to domain-specific personality traits, inter- and intrapersonal variables (represented in figure 1). A typical attribute of the development of students as co-cognitive variables is an increasing motivation. This impact may be explained by an appropriate learning pace ("A similar pace of learning increases motivation", translated from German), the interest and the joy of working on projects ("Projects that are of interest to children are motivation-enhancing", translated from German) or the individually appropriate task level ("Great motivation for students, since they are not underwhelmed. They work according to their level of performance", translated from German) to name only a few. In addition, it might seem that there is also a significant impact on the self-confidence of participating pupils, the willingness to make an effort and the interest in mathematical contexts to name only a few personality traits affected by participating in a revolving door model. The questionnaires indicate also an impact on intrapersonal variables, such as affective and emotional consequences, and on interpersonal variables, such as the improvement of social interaction with other students or the development of (self-reliant) learning strategies. In addition, the questionnaire indicates also an improvement of cognitive variables. Apart from general variables (such as "the enhancement of knowledge", translated from German), these cognitive variables include both indicators of mathematical giftedness, such as "structuring" or "originality in mathematical contexts", and criteria focusing on mathematical general education, such as "communication about mathematical contents".

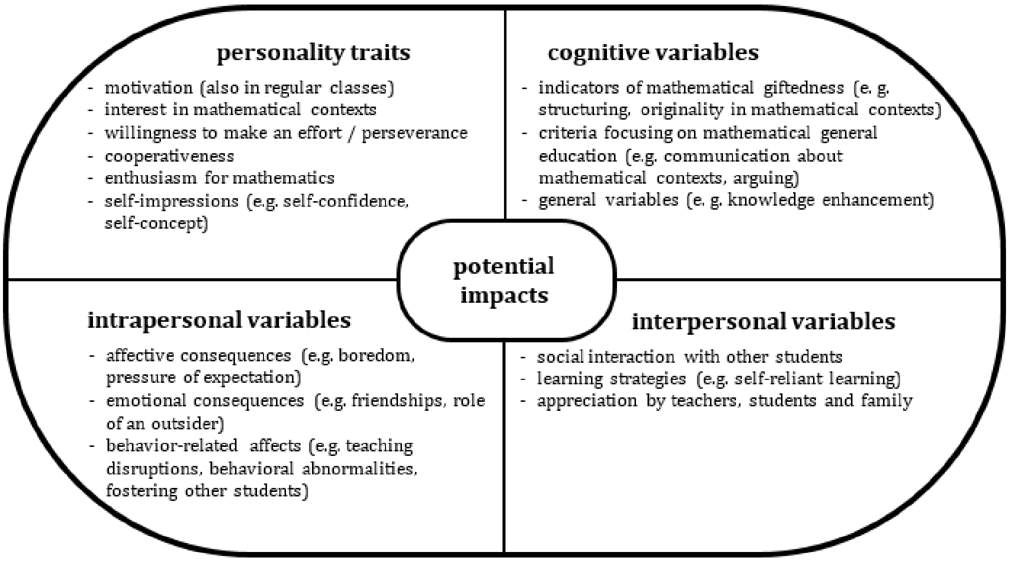

Figure 1: Explorations of the potential impacts (first results from questionnaires).

PERSPECTIVES

Further results of both the questionnaire and the interview study can be expected for the end of 2019; afterwards the case studies will follow. The first results of the questionnaire study seem to promise exciting observable impacts of taking part in a revolving door model, especially on the development of students as co-cognitive variables. The anticipated studies, however, will allow deeper clarifications and more differentiated insights into individual processes against the background of different approaches of revolving door models. The presented dissertation project is part of the nationwide initiative 'LemaS' aiming to develop both guiding principles and practical concepts for supporting achievement adaptively (for first impressions, see Benölken, Käpnick, Auhagen & Schreiber, 2019). Thus, based on the results of these investigations the dissertation project aims to give practical consequences for the implementation of revolving door models at schools as one concept for supporting gifted pupils.

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CREATIVE PROCESSES OF FIRST GRADERS WORKING ON ARITHEMTIC OPEN TASKS

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Key words: Mathematical creativity, first graders, open tasks, arithmetic.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

In the field of mathematics education, "the word creativity is 'fuzzy' and lends itself to a variety of interpretations" (Sriraman, 2005, p. 20). Therefore, it is necessary to define creativity more precisely in a content-specific way to make this term more adaptable for mathematics instruction. In this research study, I want to answer the following three research questions that are afterwards related to the theoretical background:

1. How diverse are creative processes of first graders working on open tasks?

In line with Kwon, Park, and Park (2006) this research focuses on students' "flexible problem-solving abilities" (Kwon et al., 2006, p. 52). Children act creatively when showing flexible competences by finding not only one but various differing solutions to an *open task.* Based on collective attributes (Levenson, Swisa, & Tabach, 2018), these creativity-enhancing tasks are characterized by the quantitative and especially qualitative differences of the solution spaces of different children. Like many other research studies (e.g. Leikin, 2009; Silver, 1997), I use the categories from Torrance (1966) to characterize the students' creative process. I aspire to illustrate the quality of the creative process and do not aim to rate their mathematical creativity. Therefore, a *mathematical creative process* in this research study is defined as the first graders' ability to find various solutions to an open task (*fluency*) that are generated by the use of different changes of ideas (*flexibility*), which are described and structured (*elaboration*) with the aim to find additional solutions (*originality*).

2. In what ways do cognitive and metacognitive instructional prompts support children to show an individual creative process?

In order to foster children's ability to show creative processes, the children can be supported by *instructional prompts*. Prompts are "recall and/or performance aids, which vary from general questions to explicit executions instructions" (Bannert, 2009, p. 139). Based on research results, the combination of *cognitive prompts* and *metacognitive prompts* has a high effectiveness on supporting students. In this study, cognitive prompts support the children to become more aware of the different and maybe up to this point unused possibilities to generate various solutions to an arithmetic open task. Furthermore, metacognitive prompts tend to support children's monitoring and controlling when working on these tasks.

3. Which conditions influence the mathematical creative process of first graders?

In contrast to many definitions that link mathematical creativity to giftedness (e.g. Sriraman, 2005) or advanced mathematical thinking (e.g. Ervynck, 1991), I assume that every primary school child has the ability to be a creative person. The term of *relative creativity* in the work with children is suggested, which means that the creativity is compared within a specific peer group. That is why every student can be creative and thus shows a creative process.

DESIGN, MATERIALS, AND PROCEDURE

This study used the *teaching experiment* methodology (Steffe & Thompson, 2000) to observe and illustrate the creative processes of first graders working on two arithmetic open tasks and the support of instructional prompts. To build a generalizable model of the creative process a sampling procedure was conducted. Thus, a *mixed method* design was used.

Quantitative study: As a sampling procedure, in March 2019 about 80 first graders from two municipal primary schools in Germany, which use different math books, were ranked by two standardized tests: first the *CFT 1-R* was used to analyze their intellectual abilities especially when problem solving, second the *MBK 1+* tested their mathematical skills.

Qualitative study: Based on the quantitative test results 18 students participated in two individual interviews in June 2019, in which they worked on two different but structurally similar arithmetic open tasks, e.g. "*Find different tasks with the number 4".* This two teaching episodes were structured through a guideline. The children got blank filling cards to write one solution on each card, arrange them on the table and explain their arrangement to me. Moreover, I gave adaptively "just-in-time-prompts" (Bannert, 2009, p. 142).

To illustrate the creative processes, I developed an analytical tool in a pilot study. In a category-lead analysis, a catalogue of categories for the changes of ideas children can use in their solutions, and categories for the supportive level of the prompts were constituted. Thus, an *individual creativity pattern,* which I represent graphically in a sort of a sequential tree diagram, can be analyzed for each first grader's creative process. I will present and discuss an exemplary first grader's creativity pattern from this study on the poster.

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MATHEMATICAL INTERACTIONS WITH GIFTED ADOLESCENTS

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Abstract. *Students who demonstrate interest in mathematics pose fascinating, and intellectually challenging, questions to each other and their teachers. As teachers and teacher educators, we make pedagogical decisions based on a quick recall of relevant content knowledge. Applying prior research and interdisciplinary literature, I reflect on my own experiences as a teacher of mathematically interested middle-grade students to determine how content knowledge and gifted education inform decisions to pursue intellectually stimulating conversations.*

Key words: Middle Grades education, Specialized Content Knowledge, Statistics

Mathematics classrooms involve many interactions with a variety of members in the room. Nearly all interactions are in the form of teacher to student, student to student, and student to teacher. With support from a teacher, students construct their own mathematical understandings. However, what happens in a classroom when students need assistance in their work a teacher did not plan to cover? Furthermore, students identified mathematically gifted, or mathematically motivated may pose questions or comments to challenge meaningfully a teacher's own mathematical understandings. Applying research questions from my previous work (Fox, 2011), I want to investigate in this study: "In what ways does a teacher apply a specialized mathematical knowledge for teaching when presented with an unanticipated student question?" (p. 8).

In mathematics education writings in the United States, researchers raise concern and pose questions regarding use of the phrase mathematically gifted (Boaler, 2016). Because the phrase mathematically gifted often suggest an implied elitism about the students served by mathematically gifted programs, I will use different language to describe the students. The students in this study chose to participate in an enrichment experience during the summer. Students were selected based on a combination of teacher recommendation and standardized test performance.

Although the literature in mathematics education provides ample description of mathematically interested students, work in science education drives part of this study forward. Park and Oliver (2009) identified moments when secondary-level students in accelerated science courses posed questions exceeded the teacher's level of immediately accessible knowledge. Teachers in this study met these challenging questions by engaging in research outside of class, returning to a future class session with the answer to the student's question.

One of the bases of the work presented in this paper is the special combination of mathematics content knowledge and pedagogical knowledge of working with students identified as gifted. Ball, Thames, and Phelps (2008) famous work on Mathematical Knowledge for Teaching includes six components of the combination of mathematical knowledge and pedagogical knowledge. Mason and Spence (1999) illustrated a sudden application of content knowledge with a flash of lightning. When necessary, teachers can recall necessary information to support the students' development of the mathematical content to provide insight quickly to answer a student's question or support students' ongoing discussions.

Rowland and colleagues' (2005) Knowledge Quartet identified four components of a pre-service primary teachers' content knowledge and applications toward the practice of teaching. The component most important in this study is the fourth component, Contingencies, are those moments teachers cannot prepare for ahead of time. However, the pedagogical decisions teachers make are informed by other components of their content knowledge.

METHOD

The mathematically interested students enrolled in this study enrolled in an intensive three-week summer enrichment course in mathematics, particularly in the domains of Data Analysis and Probability. In this classroom across 15 days, the same 12 students were in the same mathematics classroom for a total of 85 hours.

This study uses several methods used in a previous study (Fox, 2011). In that study, I used audio recording devices to capture classroom interactions between students and the teacher. I served as a non-participating observer in two accelerated secondary mathematics classes. During an observation of a classroom session, I made notes to identify key moments for additional reflection by the teacher and myself, the researcher. Two times during the observation, I interviewed the teacher about key moments to identify the teacher's pedagogical move and mathematical backing to implement the move. Serving dual roles of researcher and teacher in the study presented in this paper, I used reflective field notes as a substitute for the interview.

Action research is a key modification to the original study (Fox, 2011). Serving dual roles as a teacher and researcher, I felt more confident in my identification of a student comment rising to the level of challenging question, one of the four main findings in the original study. In order to determine if such a comment in this newer study was worthy of additional consideration, I used three criteria of a Mathematically Important Pedagogical Opportunity from Leatham and colleagues (2011): mathematically important, a pedagogical opportunity, and represents evidence of student thinking.

Acknowledging a surprising question as challenging is designed to be a pedagogical tool, a transition from the I-R-E discourse pattern to a reflective toss (van Zee & Minstrell, 2007). This transition allows for students to take ownership of the mathematical content, instead of having the teacher always verifying the validity of a student's comment or conjecture. If a student's comment satisfied Leatham and colleagues' criteria, I could apply patterns I developed in a previous work (Fox, 2011): pursue the student's in isolation or include the student's comment while maintain the planned progression of the lesson. With no formal marks assigned or required curriculum prescribed, I, as the teacher, could take time from my plan to address students' questions fully I did not originally plan to discuss.

RESULTS

In a previous paper (Fox, 2014), I discussed how mathematically interested these middle-grades students provided surprising answers to a contextual problem. When the students offered a solution I did not anticipate, I probed the student's explanation to gain additional insight as to how and why the student arrived at his/her solution. In some instances, the student could not readily identify why he/she determined the solution to the problem.

Because the students in the study were mathematically motivated and young, I conjectured they had not had many chances in their previous schooling to explain why they solved a problem the way they did.

An additional moment from the same classroom to discuss in this poster is a student question about measures of spread. I developed a progression of activities for students to compute the measures of spread, based on the recommendations of the GAISE report (Franklin et al, 2005): mean absolute deviation–suggested to be taught at the middle-grades level–before discussing standard deviation to the same mathematically interested students. The authors of the GAISE report suggest teachers present standard deviation to students at the secondary level, but I wanted to present the material to these mathematically interested middle-grades students both to pique their interest and to offer students a comparison between these two measures. Giving the students permission to ask questions whenever they wanted, one student posed an interesting question: how to determine these measures of spread when numbers are negative.

Having only presented positive-valued data to students, I had not thought about this question before. I quickly determined the students question fulfilled Leatham and colleagues' (2011) criteria. From the mathematical perspective, I thought answering the student's question would lead to an interesting pursuit about computations and the conceptual understanding of measures of spread. From a pedagogical perspective, I thought pursuing a question a student posed from observing my work was worthy of additional exploration. Furthermore, the classroom context afforded me ample time to acknowledge and wrestle with an unanticipated question. When a student posed a question I did not anticipate, I had the opportunity, in terms of time in class, to probe the mathematical understandings connected to a challenging question.

Applying my work with unanticipated questions a teacher did not have an immediate answer for (Fox, 2011), I modified the trajectory of my lesson to incorporate the student's comment into the planned lesson by answering the question simultaneously while moving forward with my planned lesson. I took the negative number the student provided and included other positive values. While moving forward in the computations, I thought of a connection between this computation and a conceptual understanding of distance these students would see later in their mathematical careers. Seeing how the current concept addressed future ideas felt like, in my own reflection, an application of the Horizon Content Knowledge described in Ball, Thames, and Phelps (2008).

One decision I made as teacher while thinking about my second role as researcher was not to discuss transforming the data to all positive values. From a mathematical perspective, adding a large enough value to all the numbers the student provided would not have impacted the average distance each value was from the mean. However, I immediately thought of the implications of teaching the students would investigate later in their mathematical careers, transformation of functions. Because my primary goal in working with these students was to enrich the school mathematics curriculum, and not accelerate the traditional mathematics curriculum found in most districts and states in the United States, I tried to steer clear of non-statistics topics and concepts in the class, particularly any content related to algebra. Teaching algebra to middle-grades students—like teaching calculus to high-school students—are two controversial ideas within the teaching of mathematically interested students in the United States. Not knowing where what the resolution of the question would be, I took intellectual and pedagogical risks in facilitating the discussion.

CONCLUSIONS

This poster describes additional interactions between student and teacher in this study. Building on the work I—and other researchers across other disciplines—did with students who demonstrate interest and aptitude in subject content, this report highlights one of the challenges facing teachers of mathematically creative and gifted students. Because these students can organize and communicate mathematics differently, teachers of these students must possess a flexible and easily accessible knowledge base and set of pedagogical practices to support the mathematical development of these students. Additional work can highlight how these challenges require specialized knowledge beyond the expectation of a typical teacher in a standard mathematics classroom.

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DIFFERENTIATED INSTRUCTION USING LEARNING MANAGEMENT SYSTEMS IN UPPER SECONDARY SCHOOL AND UNIVERSITY LEVEL – A RESEARCH PROPOSAL

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Abstract. *There is a need to develop an infrastructure to support and maintain teaching that both challenge all students at their knowledge level and open up the possibility to applying mathematical knowledge in innovative and creative ways. Learning management systems (LMS) are widely used throughout the Swedish school system. However, recent studies shows that few teachers use this resource for teaching development, i.e., using LMS as an instrument to improve forthcoming lessons. In this poster a research proposal is outlined. The aim is to explore how LMS can be used as an instrument for differentiated instruction throughout the intertwined processes of planning, teaching, studying, follow up and assessment.*

Keywords: Differentiated instruction, Learning Management System

INTRODUCTION

Swedish teachers in upper secondary school and at the university level tend to strictly follow a textbook (cf. Boesen et al., 2014; Lithner, 2007). From this it follows that students in a class often interact with the same items, resulting in stress for some, and boredom for others. Further, van Langen and Dekkers (2005) found that Sweden in comparison with 11 other OECD countries had the highest drop-out rate on STEM courses at the university level. Only 48% of the students passed the courses. This high drop-out rate is still a problem. Hence, there is a need to improve teaching to improve every student's chance of applying mathematical knowledge in innovative and creative ways.

Teachers are interested in developing differentiated instructions (cf. Mellroth, 2018), but recent studies show that schools are not organized in a way that supports the planning and follow-up of teaching (Nordgren et al., 2019). There is a need to develop supportive structures in schools on all levels to improve mathematics teaching. Specifically, there is a need for strategies for differentiated instruction challenging students at their knowlege level. LMS are online environments that enable communication between teachers and students, 95% of all upper secondary schools and all universities use LMS in Sweden. Therefore, using LMS as a pedagogical instrument, rather than merely an administrative tool, may be a fruitful way to use already existing infrastructure. The research question guiding our research proposal is: How can LMS be used as an effective instrument for improving mathematics teaching? Especially we focus our study on how the LMS may be used to differentiate instruction regarding mathematical content.

AIM

The aim is to develop strategies for using LMS to orchestrate differentiation of mathematical items and activities. The administrative aspect of the LMS regarding students' results and continuous assessment will be put to work as resources for teaching development, and differentiated instruction.

The long-term goal is to develop infrastructure that supports teaching which addresses each student's capability and which nurtures innovative and creative ways of thinking. This may lead to more students passing university mathematics and less stressed or bored upper secondary students.

DESIGN OF STUDY

In this project, teachers and researchers, from both upper secondary school and the University, form an epistemic community where teachers' professional knowledge (from both school levels) contributes with a different perspective (Ruthven & Goodchild, 2008). The research will be conducted within design research cycles (McKenney & Reeves, 2018). Students' outcomes and the development of items and activities supporting all students at their knowledge level will be in focus. We will, for example, study how LMS can be used to effectively follow and respond on students' work. Formative assessment on a digital platform will be in focus to differentiate instructions for groups of students. The hypothetical learning trajectory and design principles will be outlined from both theoretical and practice related knowledge (e.g., Lithner, 2007; Mellroth, 2018; Sheffield, 2003). The affordances and constraints using LMS as an instrument for teaching development both in upper secondary, and university level will be investigated and described.

FINAL COMMENTS

At the MCG conference we will discuss our research proposal, and we welcome comments and suggestions from the participants to improve and sharpen the design.

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RESEARCH AND DEVELOPMENT TASKS WITHIN THE FRAMEWORK OF THE PRIMA-PROJEKT IN HAMBURG

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Abstract. *PriMa is a project aiming amongst others on the promoting of mathematical gifted pupils of elementary schools in Hamburg. Whilst working with the mathematical gifted pupils within the project questions regularly occur. These questions lead to research and development projects within PriMa. Current research projects are presented within this paper.*

Key words: mathematical gifted children, research projects, development projects

THE PRIMA-PROJEKT

PriMa is a cooperation program of the University of Hamburg, Hamburg Authority for Schools and Vocational Education (BSB) and William-Stern-Society (WSG). The PriMa program consists of different subprograms. One of these programs is a university project aiming to support mathematically gifted children from the third grade onwards.

Since 20 years, the University of Hamburg has been conducting a three-stage talent search once a year. On a weekend, interested third graders can try out whether they have interest and joy in solving specially developed progressive investigating problems. The meeting is called *math meeting for math fans* (in German "*Mathe-Treff* für *Mathe-Fans*"). This is followed by a mathematics test and an intelligence test. Based on the tests results, 50 children will be offered a place in a group in the university project.

All other children who have gone through the talent search till the end are offered the opportunity to participate in a so-called math circle, provided at different primary schools and led by mathematics teachers (Nolte, 2015).

ELEMENTS OF THE CONCEPT OF THE UNIVERSITY PROJECT

The culture of each subject must be regarded as an essential aspect for the promotion of students. Enculturation, whose significance for learning processes is described by Brown et al. (1989: 34, after (Lave & Wenger, 1991)), can already be found in Krutetskii, who speaks of "mathematical cast of mind" in relation to the development of mathematical competences. Enculturation includes growing into norms, language, behavior and typical activities of the respective subject. Although it is the same subject, mathematics teaching at school follows different rules than those of the university project of PriMa, which is growing into research-oriented mathematic learning in a high complexity.

Within the university project of PriMa, the children gain their first age-appropriate access to mathematical theory development processes in complex problem fields . Particularly when entering a problem field, at primary grade level they are guided in order to lead the pupils to the mathematical core of the problem, and to avoid getting lost in a variety of possible questions.

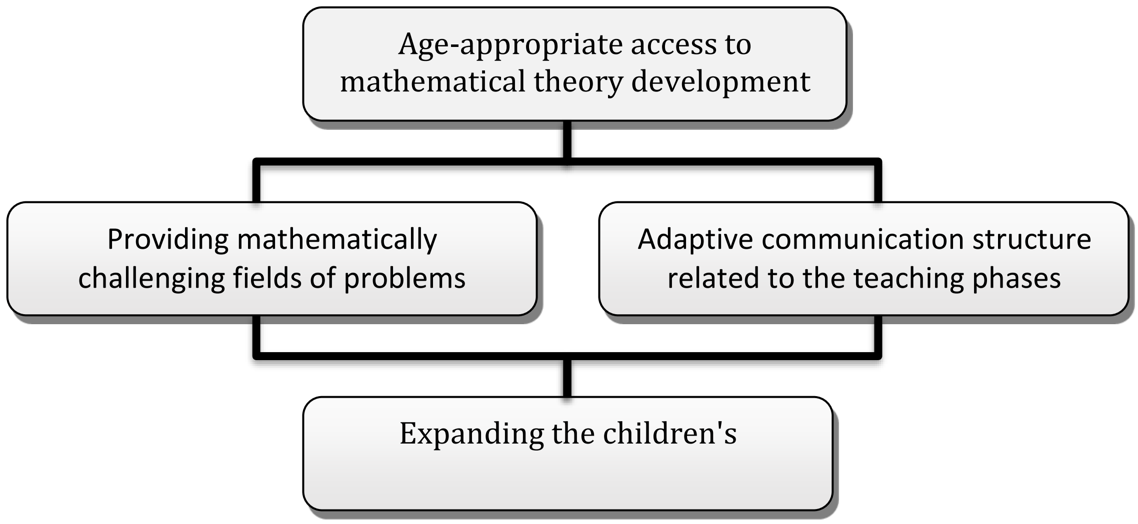

Figure 2: elements of the concept of the university project

At the same time, the children are instructed to communicate their thoughts in mathematical language. Due to the often very complex trains of thought of the children in the university project, this is a challenge for the pupils (as they often have only rare opportunity to practice this in school at this level), as well as for the tutors. Thus, tutors are intensively trained for interaction and communication processes with the children. The aim of the project is to expand the children's subjective action space (see Fig. 1).

CURRENT RESEARCH AND DEVELOPMENT PROJECTS

Currently there are mainly three research projects within the Prima-Project:

Development and Evaluation of an observation instrument for diagnostic (Kirsten Pamperien)

Evaluating the performance of children cannot be done solely result-oriented, because the kind of problem-solving strategies used by children are decisive for the quality of the performance. Thus, current research deals with the development and evaluation of observation scales, which items are based on patterns of action in combination with higher order thinking skills, specific to the particular problem.

An essential competence of teachers is observing and accompanying learning processes. However, especially process-oriented diagnostics in complex problem areas, i.e. keeping an eye on the diversity of possible solutions, is a challenging task. An observation instrument in the sense of a task-specific checklist may help (Heller et al., 2005).

On the one hand, we use the instrument as a supplemental screening tool within the trial lessons at the Mathe-Treff in order to obtain even better initial indications of a mathematical potential. On the other hand, we investigate whether the instrument can be used in regular lessons at school. The aim is a better recognition and assessment of student's development in using higher order thinking skills. Prerequisite is an intensive teacher or tutor training in observing problem solving processes. (Nolte, 2012; Pamperien, 2004; Nolte & Pamperien, 2017).

Usage of observation tools by prospective teachers (Marianne Nolte, Merve Bektaş, Laura Moczkuhn)

The study examines how well prospective teachers are able to identify action patterns and cognitive components of problem solving processes during teaching small groups of children while using observation scales. These are developed for specific problems. First results point to the necessary familiarity with the mathematical background. They also refer to the importance of emotional aspects on the side of prospective teachers as well as on the side of the students. Furthermore, they show how challenging it is to accompany even a small group.

Concept development for the extension of the Uni-Project from grade 7 (Katrin Vorhölter)

The university project is planned to continue after class 7 in the near future. For the development of a concept, it must be ensured, as shown above, that it is oriented to the children's level of development, their Subjective Action Space (Ziegler et al. 2013) and the mathematics school curriculum.

On the one hand, this has an impact on the type of formulating problem fields, which stepwise have to be less pre-structured. On the other hand, the way of communication changes due to a higher grade of abstraction that goes along with algebraisation and proofing in more formal way. However, the aim will be the same: expanding the students' Subjective Action Space while working on complex fields of problems.

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